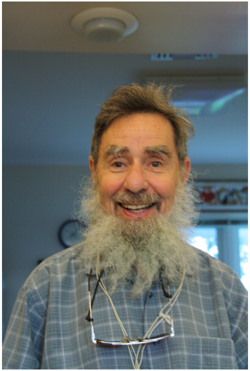
$\theta(p_c)$?

à partir de travaux avec H. Duminil-Copin and V. Sidoravicius

Vincent Tassion

ENS DE LYON

JPS, 10 avril 2014



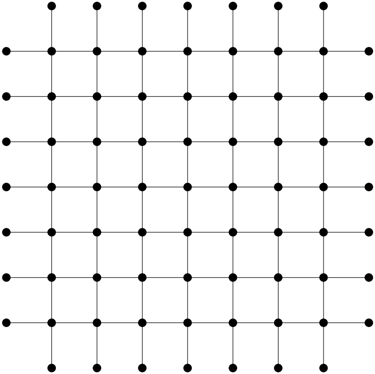
Quite apart from the fact that percolation theory had its origin in an honest applied problem (see Hammersley and Welsh (1980)), it is a source of fascinating problems of the best kind a mathematician can wish for: problems which are easy to state with a minimum of preparation, but whose solutions are (apparently) difficult and require new methods.

Harry Kesten

Épisode 1 : Percolation de Bernoulli

Percolation de Bernoulli

$G = (V, E)$, $p \in [0, 1]$.



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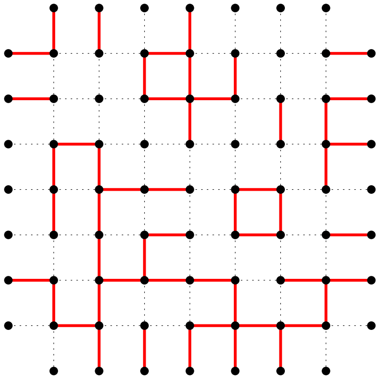

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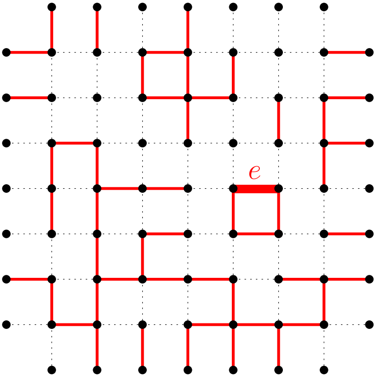

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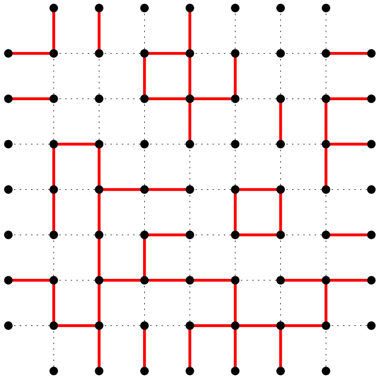

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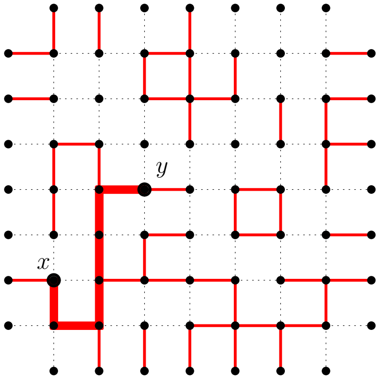

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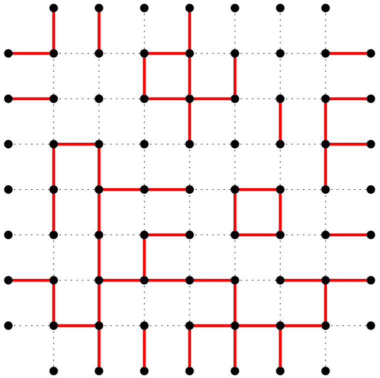

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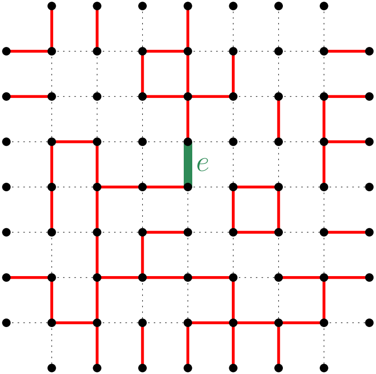

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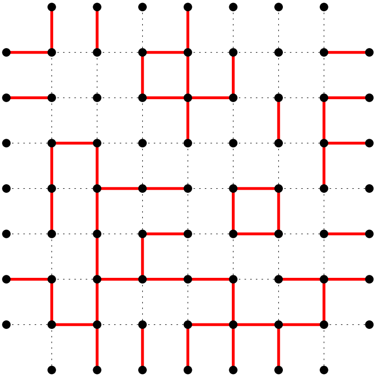

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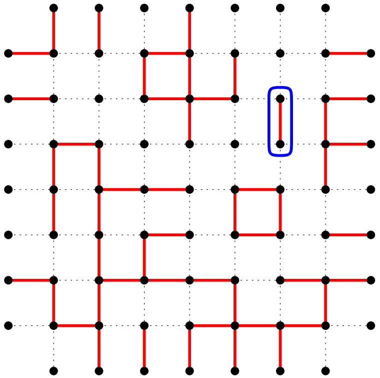

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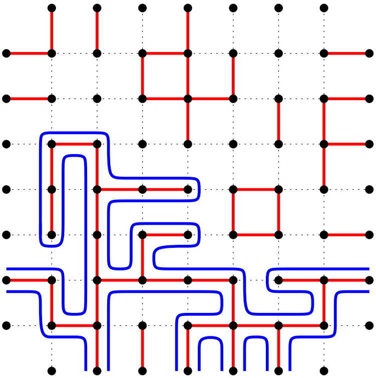

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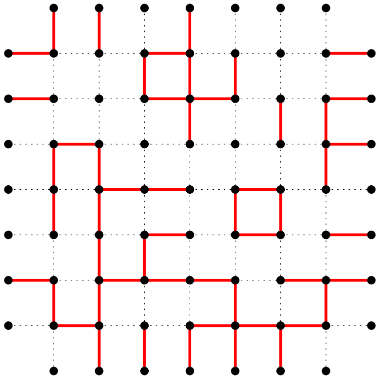

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Définition de la transition de phase

Soit G un graphe infini.

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$$\theta(p) = \mathbf{P}_p^G[0 \leftrightarrow \infty]$$

$0 \leftrightarrow \infty$: le cluster de 0 est infini.

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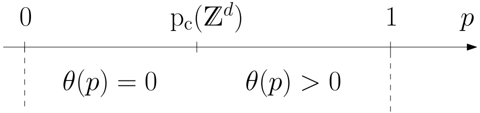
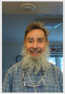


Que vaut p_c ?

Percolation sur le réseau carré

Théorème [Kesten, 1980] 

$$p_c(\mathbb{Z}^2) = \frac{1}{2}$$

$$\theta(p_c)\,?$$

Théorème [Russo, 1980]

Pour $G = \mathbb{Z}^2$,

$$\theta\left(p_c(G)\right) = 0.$$

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Conjecture

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LE GRAAL !

Pour $G = \mathbb{Z}^3$,

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Commençons par une tranche

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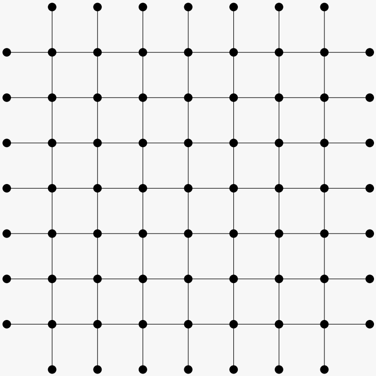
Motivation :

Théorème [Grimmett-Marstrand (1990)]

$$\lim_{k\to\infty} p_c\left(\mathbb{Z}^2 \times \{0, \ldots, k\}\right) = p_c\left(\mathbb{Z}^3\right).$$

Épisode 2 : Percolation FK sur le réseau carré $\mathbb{Z}^2$

Percolation FK sur Λ_n



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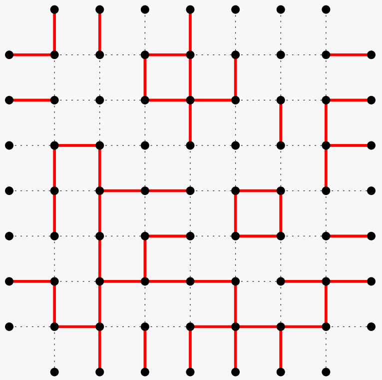

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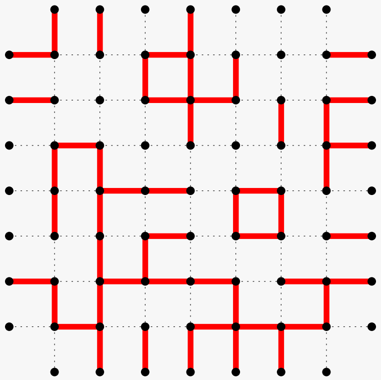

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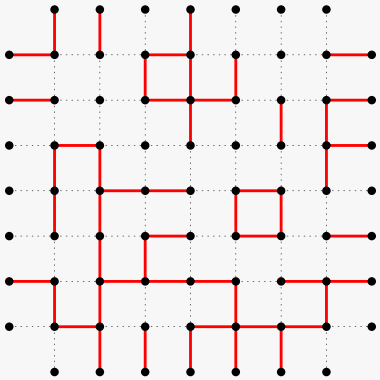

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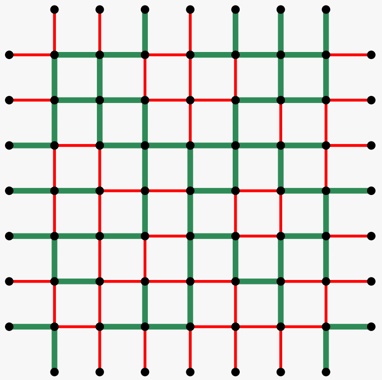

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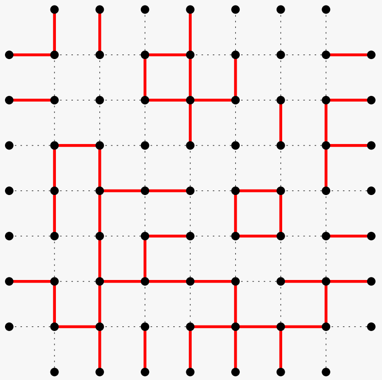

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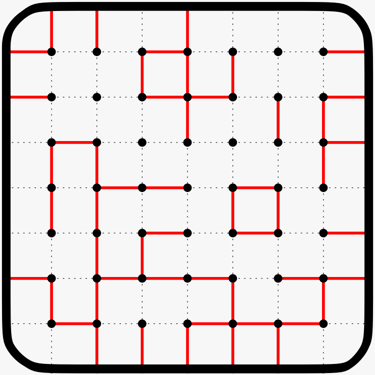

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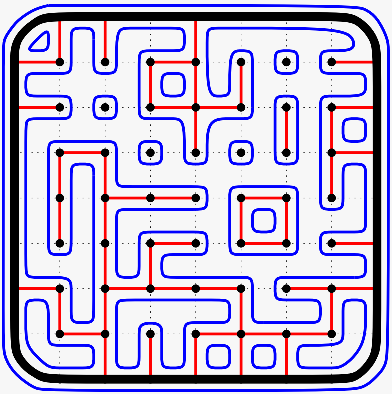

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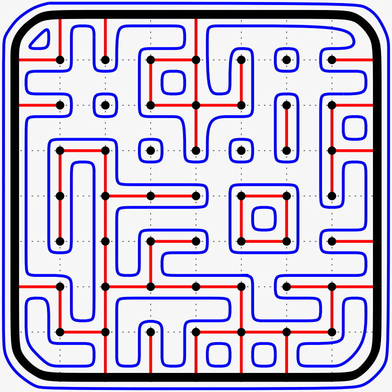

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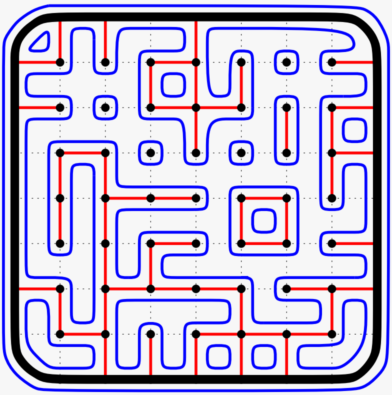

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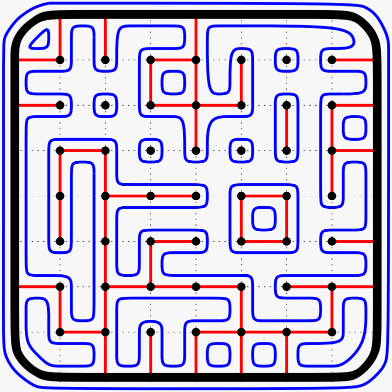

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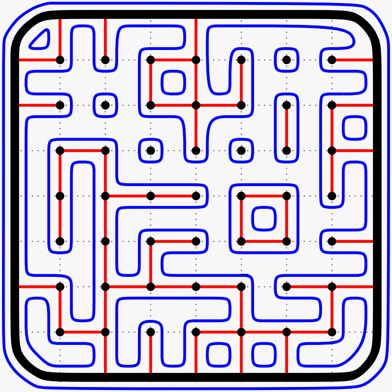

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Percolation FK sur $\mathbb{Z}^2$

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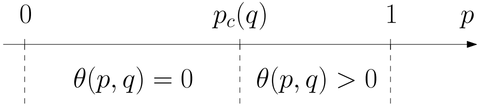

Que vaut p_c ?

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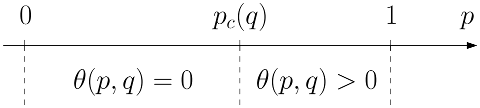

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Pour tout $q \geqslant 1$, we have

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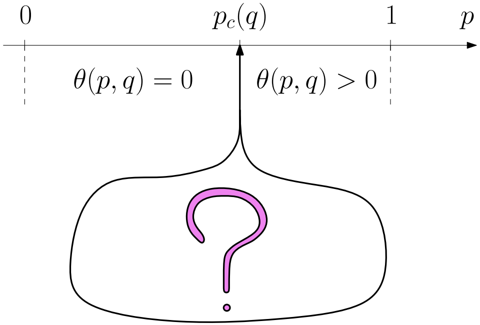

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Conjecture [Baxter, 1978]

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Si $q \leqslant 4$, alors
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Si $q > 4$, alors
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Comportement critique

Théorème

Si $q \leqslant 4$, alors

$$\theta(\mathrm{p_c}(q), q) = 0.$$

Déjà connus :

- Cas $q = 1$ (percolation de Bernoulli) [Russo, 1980].
- Cas $q = 2$ (couplage avec le modèle d'Ising) [Onsager 1944, Yang 1951].

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Déjà connus :

- Cas $q = 1$ (percolation de Bernoulli) [Russo, 1980].
- Cas $q = 2$ (couplage avec le modèle d'Ising) [Onsager 1944, Yang 1951].

Conjecture [Baxter, 1978]

Si $q > 4$, alors
$$\theta(\mathrm{p_c}(q), q) > 0.$$

Déjà connu :

- Cas $q > 25.92$ [Laanait, Messager, Miracle-Solé, Ruiz, Shlosman, 1991]

Épisode 3 : Modèle de Potts sur le réseau carré $\mathbb{Z}^2$

Potts model [Potts-Domb, 1951]

Definition : On se donne q couleurs $1, \ldots, q$. (1=rouge)
On colorie tout le monde en dehors de Λ_n en rouge.
Pour x dans la boîte Λ_n, on tire au hasard une couleur $\sigma(x) \in \{1, \ldots, q\}$.

$$\mathbf{P}^{1}_{\Lambda_n,T,q}[\{\sigma\}] \propto \exp\left(-\frac{1}{T}H(\sigma)\right) \quad \text{où } H(\sigma) = \sum_{\substack{z,z' \in \Lambda_n \\ |z-z'|=1}} \mathbb{1}_{\sigma(z) \neq \sigma(z')}$$

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Definition : $\theta(T) = \lim_{n\to\infty} \mathbf{P}^1_{\Lambda_n,T,q}[\sigma(0) = 1] - 1/q$.

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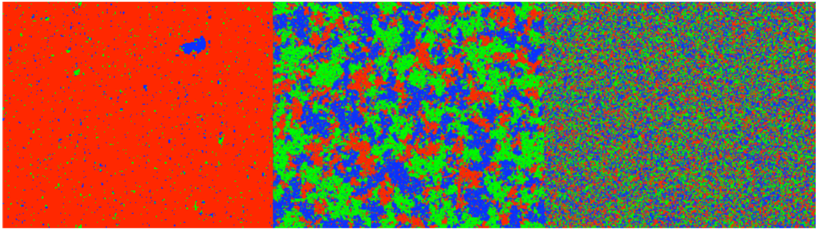
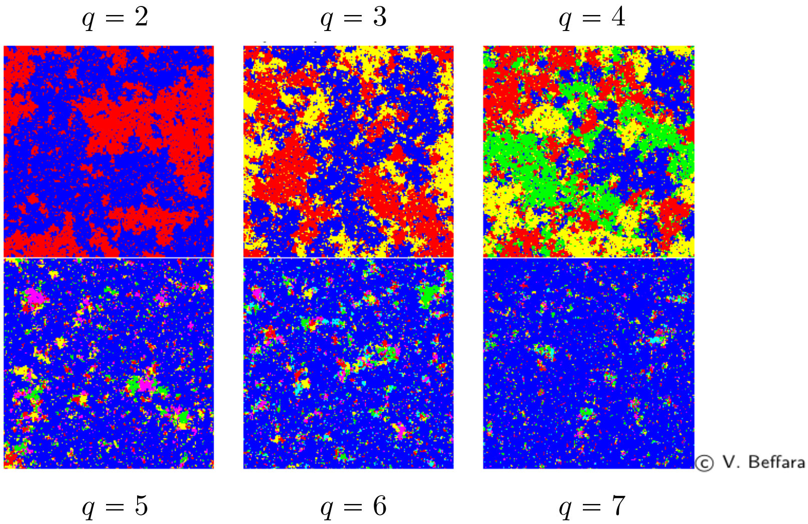
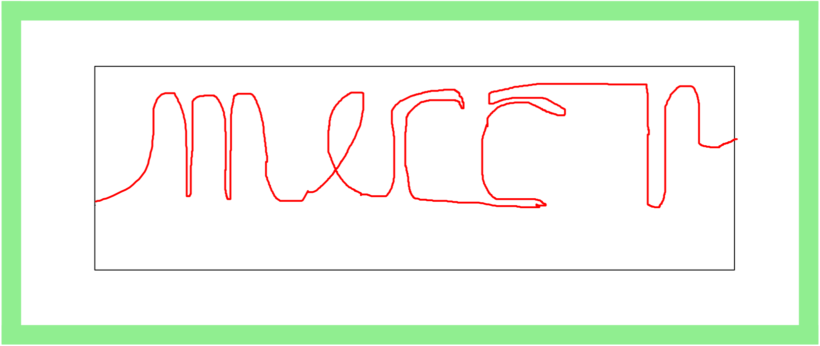


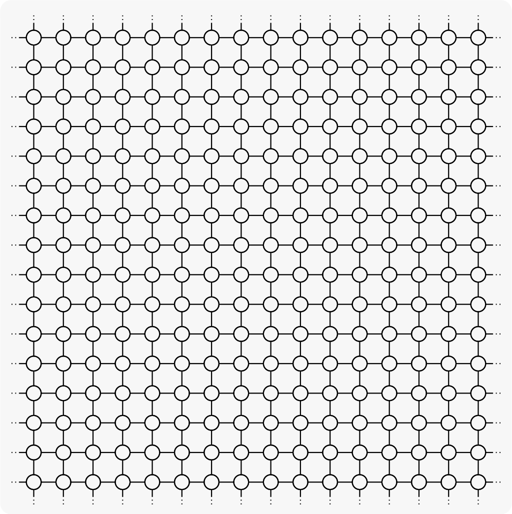
$T < T_c, \theta(T) > 0$ $\qquad$ $T = T_c$ $\qquad$ $T > T_c, \theta(T) = 0$

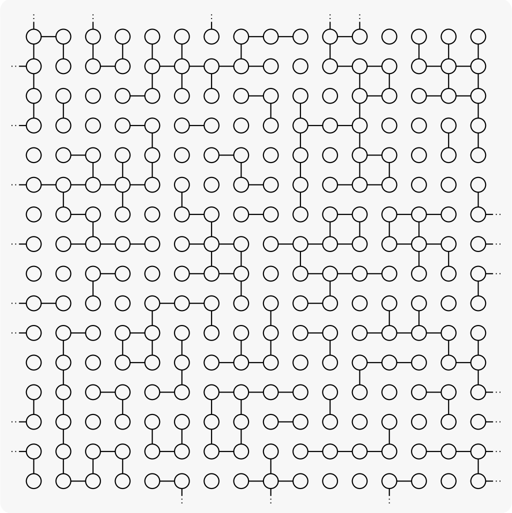
$$\theta(T_c)\,?$$

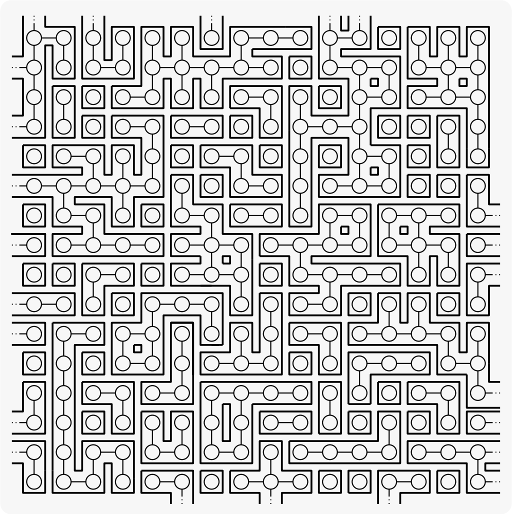
Modèle de Potts critique, $T = T_c$

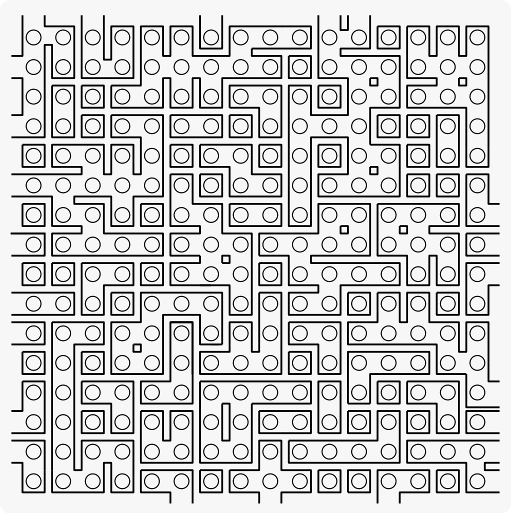


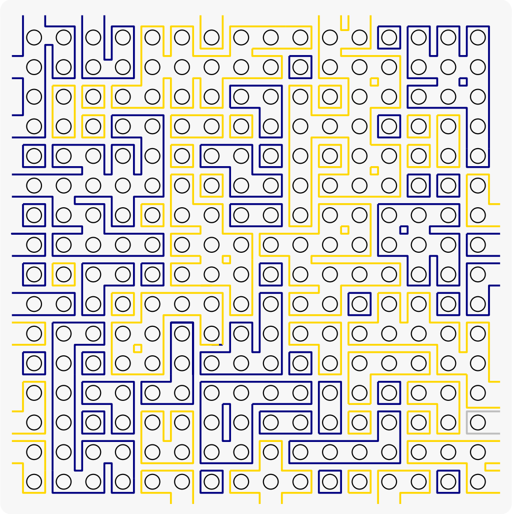


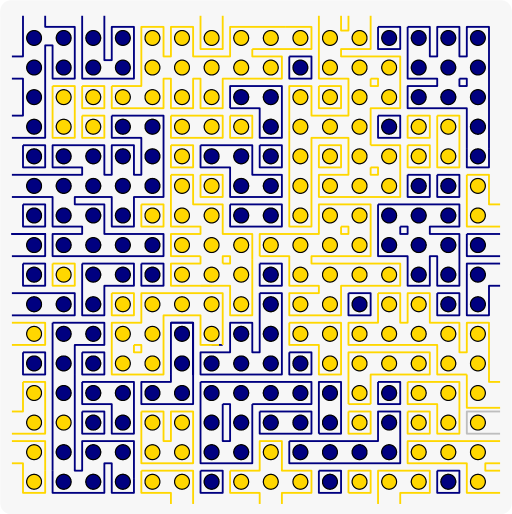


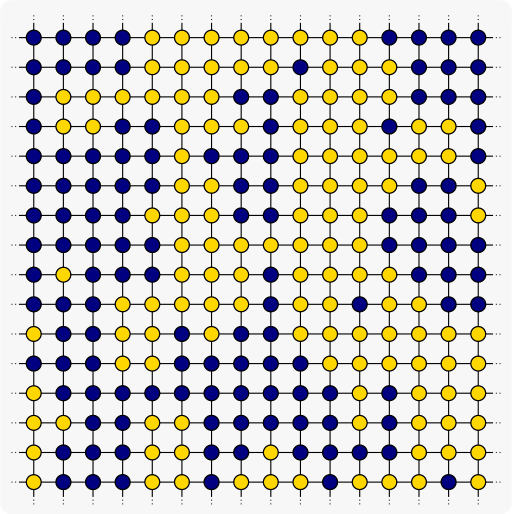




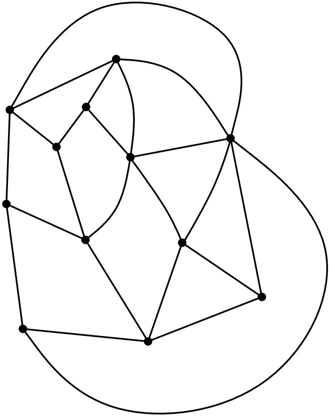






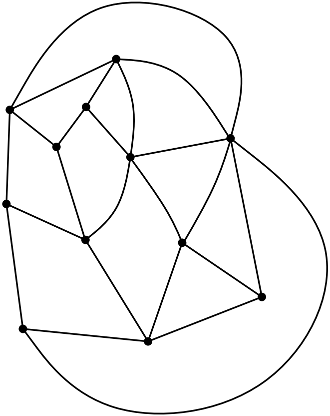


Duality for a finite Planar graph



	PRIMAL	DUAL
Graphs	$G = (V, E)$ finite planar graph	
Edges		
Configurations		
Measures		

Duality for a finite Planar graph



	PRIMAL	DUAL
Graphs	$G = (V, E)$ finite planar graph	$G^* = (V^*, E^*)$
Edges		
Configurations		
Measures		

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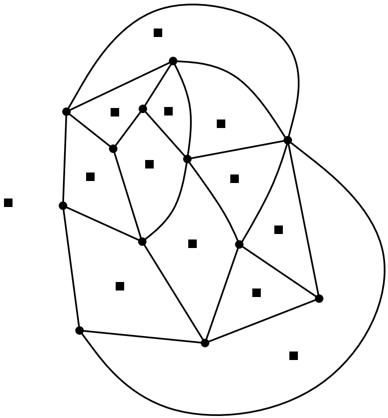

	PRIMAL	**DUAL**
Graphs	$G = (V, E)$ finite planar graph	$G^* = (V^*, E^*)$ V^*:=faces(G)
Edges		
Configurations		
Measures		

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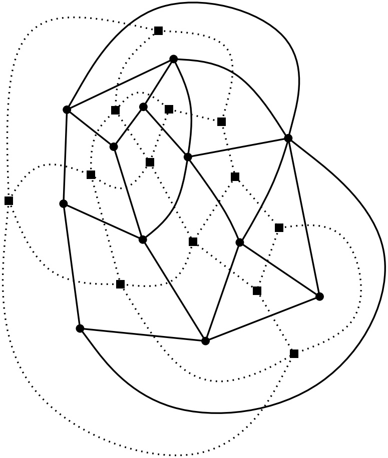

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Graphs	$G = (V, E)$ finite planar graph	$G^* = (V^*, E^*)$ V^*:=faces(G) E^*:=pairs of neighbour faces
Edges		
Configurations		
Measures		

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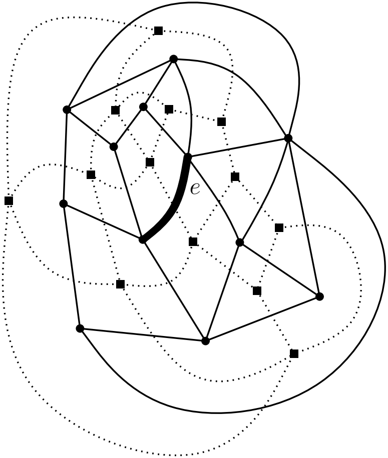

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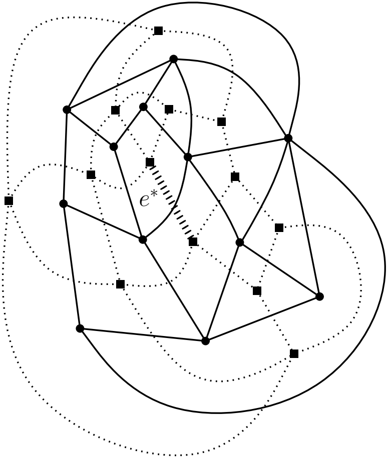
Graphs	$G = (V, E)$ finite planar graph	$G^* = (V^*, E^*)$ V^*:=faces(G) E^*:=pairs of neighbour faces
Edges	$e \in E$	
Configurations		
Measures		

Duality for a finite Planar graph



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Graphs	$G = (V, E)$ finite planar graph	$G^* = (V^*, E^*)$ V^*:=faces(G) E^*:=pairs of neighbour faces
Edges	$e \in E$	$e^* \in E^*$ unique edge of E^* crossing e.
Configurations		
Measures		

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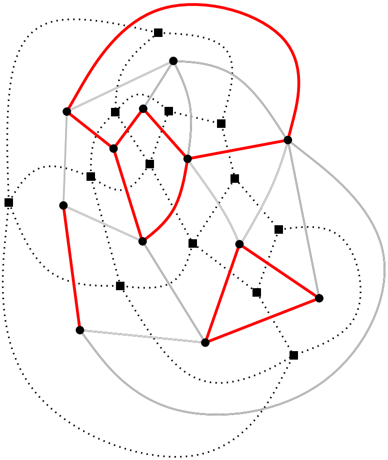

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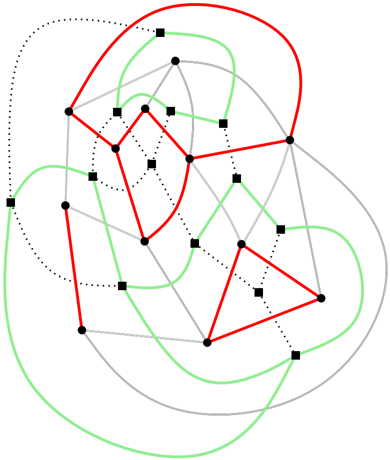
Duality for a finite Planar graph



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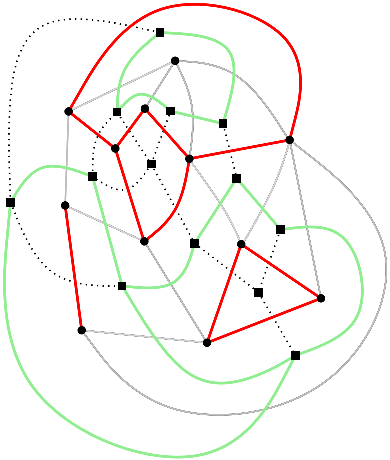
Configurations	$\omega \in \{0,1\}^E$	$\omega^* \in \{0,1\}^{E^*}$ $\omega^*(e^*) := 1 - \omega(e)$
Measures		

Duality for a finite Planar graph



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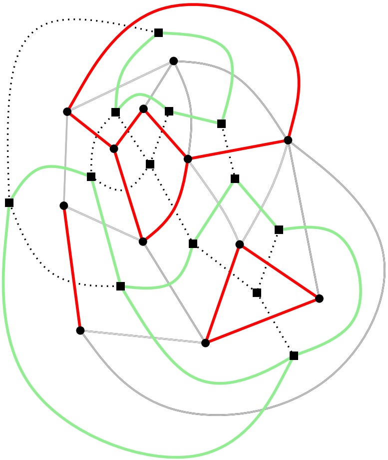
Measures	$\omega \sim FK^G_{p,q}$	$\omega^* \sim FK^{G^*}_{p^*,q}$ $\frac{p^*}{1-p^*} = \frac{q(1-p)}{p}$

Duality for a finite Planar graph



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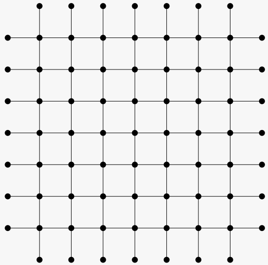
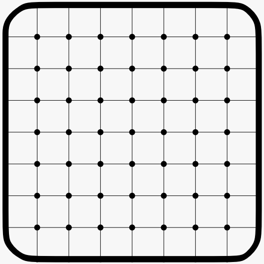
Measures $p = \frac{\sqrt{q}}{1+\sqrt{q}}$	$\omega \sim FK^G_{p,q}$	$\omega^* \sim FK^{G^*}_{p,q}$

$\Lambda_n = \{-n, \ldots, n\}^2 \subset \mathbb{Z}^2$, ω configuration on Λ_n.

Free boundary conditions



Wired boundary conditions



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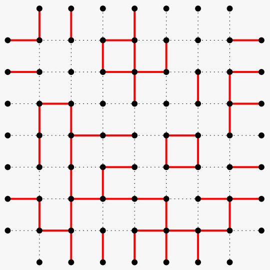

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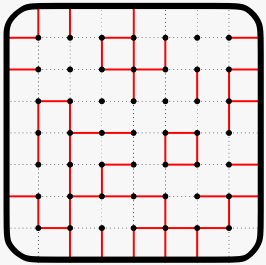

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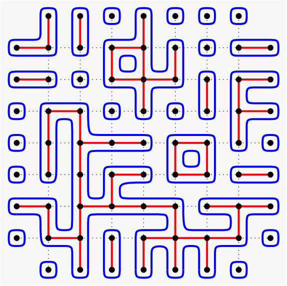
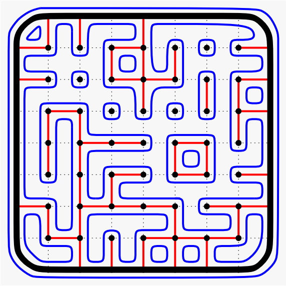


Number of clusters : $k_{\text{free}}(\omega)$.

Wired boundary conditions



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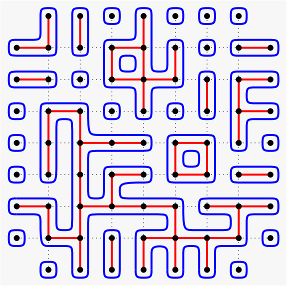
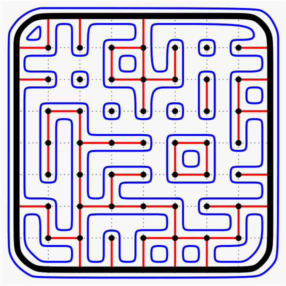


Number of clusters : $k_{\text{free}}(\omega)$.

$$\mathbf{P}^{\mathrm{f}}_{\Lambda_n,p,q}[\omega] = \frac{1}{Z}\left(\frac{p}{1-p}\right)^{\mathrm{o}(\omega)} q^{\mathbf{k}_{\mathbf{free}}(}$$

Wired boundary conditions



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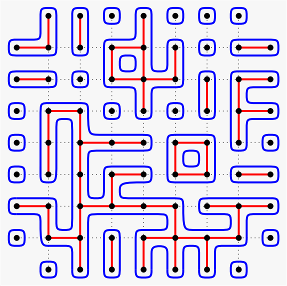

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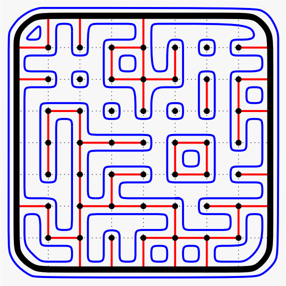

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