Density Bounds for some Degenerate Stable Driven SDEs.

Lorick Huang

Laboratoire de Probabilités et Modèles Aléatoires
Université Paris 7

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Degenerate SDE?

We consider the following degenerate stable-driven SDE:

$$\begin{aligned}
dX_t^1 &= \left(a_t^{1,1}X_t^1 + a_t^{1,2}X_t^2 + \cdots + a_t^{1,n-1}X_t^{n-1} + a_t^{1,n}X_t^n\right)dt + \sigma(t,X_{t-})dZ_t \\
dX_t^2 &= \left(a_t^{2,1}X_t^1 + a_t^{2,2}X_t^2 + \cdots + a_t^{2,n-1}X_t^{n-1} + a_t^{2,n}X_t^n\right)dt \\
dX_t^3 &= \left(a_t^{3,2}X_t^2 + \cdots + a_t^{3,n-1}X_t^{n-1} + a_t^{3,n}X_t^n\right)dt \\
&\vdots \\
dX_t^n &= \left(a_t^{n,n-1}X_t^{n-1} + a_t^{n,n}X_t^n\right)dt
\end{aligned} \tag{1}$$

with initial condition $X_0 = x \in \mathbb{R}^{nd}$, and where

- Z is an $\mathbb{R}^d$ valued symmetric α stable process ($\alpha \in (0,2)$),
- $\sigma : \mathbb{R}_+ \times \mathbb{R}^{nd} \to \mathbb{R}^d \otimes \mathbb{R}^d$, Hölder continuous, uniformly elliptic, bounded.
- $a^{i,j} : \mathbb{R}_+ \to \mathbb{R}^d \otimes \mathbb{R}^d$,
- for $x \in \mathbb{R}^{nd}$, we denote $x = (x^1, \cdots, x^n)$, with $x^i \in \mathbb{R}^d$.

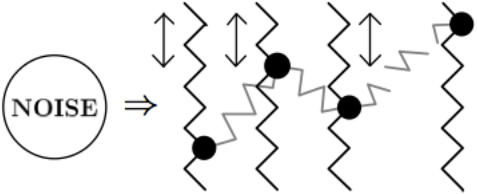
Motivation

Two examples.

- For $n = 2$, the pricing of the Asian option in jump diffusion models.

$$
\begin{aligned}
X_t^1 &= x^1 + \int_0^t a_s^1 X_s^1 ds + \int_0^t \sigma(s, X_{s-}) dZ_s \\
X_t^2 &= x^2 + \int_0^t a_t^2 X_s^1 ds
\end{aligned}
$$

- In Physics, a system of perturbated oscillators:



The Stable Process

- The Lévy measure factorizes as $\nu(dz) = C\frac{d|z|}{|z|^{1+\alpha}}\mu(d\bar{z})$ and its Fourier transform writes:

$$\mathbb{E}\left(e^{i\langle p, Z_t\rangle}\right) = \exp\left(-t\int_{S^{d-1}} |\langle p, \varsigma\rangle|^{\alpha}\mu(d\varsigma)\right).$$

- When μ the spectral measure is non degenerate, the density exists and we have the estimates:

$$c\frac{t^{-\frac{d}{\alpha}}}{\left[1+\frac{|z|}{t^{\frac{1}{\alpha}}}\right]^{d+\alpha}} \le p_{Z_t}(z) \le C\frac{t^{-\frac{d}{\alpha}}}{\left[1+\frac{|z|}{t^{\frac{1}{\alpha}}}\right]^{d+\alpha}}.$$

- When $|z| \le t^{\frac{1}{\alpha}}$, we say that the **diagonal** regime holds, and:

$$ct^{-\frac{d}{\alpha}} \le p_{Z_t}(z) \le Ct^{-\frac{d}{\alpha}}.$$

- When $|z| \ge t^{\frac{1}{\alpha}}$ we say that the **off-diagonal** regime holds, and

$$c\frac{t}{|z|^{d+\alpha}} \le p_{Z_t}(z) \le C\frac{t}{|z|^{d+\alpha}}.$$

Related works

- In the case $n = 1$ and $\alpha = 2$, the brownian estimate under uniform elliptic setting is known from Friedman 1964 [3].
- When $n = 1$, Kolokoltsov in 2000 [4] showed that for a stable diffusion:

$$c\frac{(s-t)^{-\frac{d}{\alpha}}}{\left[1+\frac{|y-x|}{(s-t)^{\frac{1}{\alpha}}}\right]^{d+\alpha}} \leq p(t,s,x,y) \leq C\frac{(s-t)^{-\frac{d}{\alpha}}}{\left[1+\frac{|y-x|}{(s-t)^{\frac{1}{\alpha}}}\right]^{d+\alpha}}.$$

- When $\alpha = 2$, Delarue and Menozzi in 2010 [2]. For the brownian chain, the estimate holds:

$$C^{-1}t^{-n^2\frac{d}{2}}\exp\left(-C\left|(\mathbb{T}^2_{s-t})^{-1}(y-R_{s,t}(x))\right|^2\right)$$

$$\leq p(t,s,x,y) \leq$$

$$Ct^{-n^2\frac{d}{2}}\exp\left(-C^{-1}\left|(\mathbb{T}^2_{s-t})^{-1}(y-R_{s,t}(x))\right|^2\right).$$

- Consider the simple case: $dX_t = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} X_t dt + \begin{pmatrix} dZ_t \\ 0 \end{pmatrix}.$
- This writes:

$$\begin{aligned} X_s^1 &= x_1 + Z_s, \\ X_s^2 &= x_2 + s x_1 + \int_0^s Z_t dt. \end{aligned}$$

- The first component is at scale $s^{1/\alpha}$, the second is at scale $s^{1+1/\alpha}$.
- We can put all the component at the same scale by normalizing by

$$\mathbb{T}_s^\alpha = \begin{pmatrix} s^{\frac{1}{\alpha}} I_d & 0 \\ 0 & s^{1+\frac{1}{\alpha}} I_d \end{pmatrix}.$$

$\rightarrow$ In the degenerate framework, the typical behavior is given by

$$\left|(\mathbb{T}_s^\alpha)^{-1}(y - R_s x)\right| \asymp \frac{|y^1 - R_s^1 x|}{s^{\frac{1}{\alpha}}} + \frac{|y^2 - R_s^2 x|}{s^{1+\frac{1}{\alpha}}}.$$

Hypotheses [H]

[H-1]: (**Hölder regularity**) $\exists H > 0,\ \eta \in (0,1],\ \forall x, y \in \mathbb{R}^{nd}$ and $\forall t \geq 0$,

$$||\sigma(t,x) - \sigma(t,y)|| \leq H|x-y|^{\eta}.$$

[H-2]: (**Non degeneracy of the spectral measure**)
$\exists \Lambda_1, \Lambda_2 \in \mathbb{R}_+^*,\ \forall u \in \mathbb{R}^d$,

$$\Lambda_1 |u|^{\alpha} \leq \int_{S^{d-1}} |\langle u, \varsigma \rangle|^{\alpha} \mu(d\varsigma) \leq \Lambda_2 |u|^{\alpha}.$$

[H-3]: (**Ellipticity**) $\exists\ \overline{c},\ \underline{c} > 0,\ \forall \xi \in \mathbb{R}^d,\ \forall z \in \mathbb{R}^{nd}$ and $\forall t \geq 0$,

$$\underline{c}|\xi|^2 \leq \langle \xi, \sigma\sigma^*(t,z)\xi \rangle \leq \overline{c}|\xi|^2.$$

[H-4]: (**Hörmander-like condition**) $\exists \overline{\alpha},\ \underline{\alpha} \in \mathbb{R}_+^*,\ \forall \xi \in \mathbb{R}^{nd}$ and $\forall t \geq 0$,
$\forall i \in [\![2, n-1]\!]$,

$$\underline{\alpha}|\xi|^2 \leq \langle a_t^{i,i-1}\xi, \xi \rangle \leq \overline{\alpha}|\xi|^2.$$

Also, for all $(i,j) \in [\![1,n]\!]^2$, $\|a_t^{i,j}\| \leq \overline{\alpha}$.

Weak Uniqueness

Theorem

*Under **[H]**, the martingale problem associated with the generator of the degenerate equation (1) admits a unique solution, i.e.:* $\forall x \in \mathbb{R}^{nd}$, $\exists! \, \mathbb{P}$ *probability measure on* $\Omega = \mathcal{D}(\mathbb{R}_+, \mathbb{R}^{nd})$ *such that*
$\forall f \in \mathcal{C}_0^{1,1}(\mathbb{R}_+ \times \mathbb{R}^{nd}, \mathbb{R})$,

$$\mathbb{P}(X_0 = x) = 1 \ \text{ and } \ f(t, X_t) - \int_0^t (\partial_u + L_u) f(u, X_u) du \ \text{ is a } \mathbb{P}\text{- martingale.}$$

Hence, weak uniqueness holds.
The associated semigroup is strong Feller.

Theorem (Density Estimates)

Under ***[H]****, the the unique weak solution of* (1) *has for every* $s > 0$ *a density. Let* $R_{s,t}$ *be the resolvent of the deterministic ODE, and let* $d = 1$ *and* $n = 2$.
For fixed $T, K > 0$, $\exists C \geq 1$, *s.t.* $\forall (x,y) \in (\mathbb{R}^2)^2$, $\forall 0 \leq t < s \leq T$,

$$p(t,s,x,y) \leq C\bar{p}_\alpha(t,s,x,y)\left(1 + \log\left(K \vee \left|(\mathbb{T}^\alpha_{s-t})^{-1}(y - R_{s,t}x)\right|\right)\right),$$

where

$$\bar{p}_\alpha(t,s,x,y) = C_\alpha \frac{(s-t)^{-\frac{1}{\alpha}-\left(1+\frac{1}{\alpha}\right)}}{\left[K + \frac{|y^1 - R^1_{s,t}x|}{(s-t)^{\frac{1}{\alpha}}} + \frac{|y^2 - R^2_{s,t}x|}{(s-t)^{1+\frac{1}{\alpha}}}\right]^{2+\alpha}}.$$

Also, when $\left|(\mathbb{T}^\alpha_{s-t})^{-1}(y - R_{s,t}x)\right| \leq K$,

$$p(t,s,x,y) \geq C^{-1}(s-t)^{-\frac{1}{\alpha}-\left(1+\frac{1}{\alpha}\right)}.$$

The Frozen process

The proxy is constructed as follows:

- let $T > 0$ arbitrary deterministic time,
- let $y \in \mathbb{R}^{nd}$ final freezing point,
- recall $R_{s,T}(y)$ satisfies $\frac{d}{ds} R_{s,T} = A_s R_{s,T}$, with $R_{T,T} = I_{nd}$ in $\mathbb{R}^{nd} \otimes \mathbb{R}^{nd}$.
- We define the frozen process as follows:

$$\begin{aligned} d\tilde{X}_s^{T,y} &= A_s \tilde{X}_s^{T,y} ds + B\sigma(s, R_{s,T}(y)) dZ_s, \\ \tilde{X}_0^{T,y} &= x, \end{aligned}$$

- We follow the deterministic system backwards.
- We set $\sigma(u, R_{u,T}(y)) = \sigma_u$.

Proposition

Fix $(t,x) \in [0,T] \times \mathbb{R}^{nd}$. The Frozen Process $\tilde{X}^{T,y}$ starting from x at time t writes:

$$\tilde{X}_s^{t,x,T,y} = R_{s,t}x + \int_t^s R_{s,u} B \sigma_u dZ_u,$$

and has a density $p_\alpha^{T,y}(t,s,x,z)$, that satisfies the bounds:

$$C^{-1}\bar{p}_\alpha(t,s,x,z) \le \tilde{p}_\alpha^{T,y}(t,s,x,z) \le C\bar{p}_\alpha(t,s,x,z),$$

where

$$\bar{p}_\alpha(t,s,x,y) = C_\alpha \frac{\det(\mathbb{T}_{s-t}^\alpha)^{-1}}{\left[K \vee \left|(\mathbb{T}_{s-t}^\alpha)^{-1}(y - R_{s,t}x)\right|\right]^{nd+\alpha}}.$$

The Parametrix Series

Assume first that $p(t,s,x,z) = \mathbb{P}(X_s \in dz | X_t = x)$ exists and is smooth. We set $\tilde{p}_\alpha(t,s,x,z) = \tilde{p}_\alpha^{s,z}(t,s,x,z)$.

$$p(t,T,x,y) - \tilde{p}(t,T,x,y) = \int_t^T d\tau \partial_\tau \int_{\mathbb{R}^{nd}} p(t,\tau,x,z)\tilde{p}(\tau,T,z,y)dz$$

Differentiating formally under the integral:

$$\begin{aligned} p(t,T,x,y) - \tilde{p}_\alpha(t,T,x,y) &= \int_t^T d\tau \int_{\mathbb{R}^{nd}} \partial_\tau p(t,\tau,x,z)\tilde{p}_\alpha(\tau,T,z,y)dz \\ &+ \int_t^T d\tau \int_{\mathbb{R}^{nd}} p(t,\tau,x,z)\partial_\tau \tilde{p}_\alpha(\tau,T,z,y)dz \end{aligned}$$

Now, from Kolmogorov's equation:

$$\partial_\tau p(t,\tau,x,z) = L_\tau^* p(t,\tau,x,z), \ \partial_\tau \tilde{p}_\alpha(\tau,T,z,y) = -\tilde{L}_\tau \tilde{p}_\alpha(\tau,T,z,y).$$

Thus,

$$p(t,T,x,y)-\tilde{p}_\alpha(t,T,x,y)=\int_t^T d\tau\int_{\mathbb{R}^{nd}} L_\tau^* p(t,\tau,x,z)\tilde{p}_\alpha(\tau,T,z,y)dz \\ -\int_t^T d\tau\int_{\mathbb{R}^{nd}} p(t,\tau,x,z)\tilde{L}_\tau\tilde{p}_\alpha(\tau,T,z,y)dz$$

Taking the adjoint in the first integral yields:

$$p(t,T,x,y)-\tilde{p}_\alpha(t,T,x,y) \\ = \int_t^T d\tau\int_{\mathbb{R}^{nd}} dz p(t,\tau,x,z)\underbrace{\left(L_\tau-\tilde{L}_\tau\right)\tilde{p}_\alpha(\tau,T,z,y)}_{H(\tau,T,z,y)}$$

That is:

$$p(t,T,x,y)=\tilde{p}_\alpha(t,T,x,y)+p\otimes H(t,T,x,y),$$

where we defined the time space convolution:

$$f\otimes g(t,T,x,y)=\int_t^T du\int_{\mathbb{R}^{nd}} dz f(t,u,x,z)g(u,T,z,y).$$

The Parametrix Series

Proposition

Let $P_{t,T}f(x) = \mathbb{E}[f(X_T)|X_t = x]$*, the transition of the solution of* (1)*, we have:*

$$P_{t,T}f(x) = \int_{\mathbb{R}^{nd}} dy \left(\sum_{r=0}^{+\infty} \tilde{p}_\alpha \otimes H^{(r)}(t,T,x,y) \right) f(y),$$

where H *is the parametrix kernel:*

$$\forall 0 \le t < T,\ (x,y) \in (\mathbb{R}^{nd})^2,\ H(t,T,x,y) := (L_t - \tilde{L}_t)\tilde{p}_\alpha^{T,y}(t,T,x,y),$$

$$L_t\varphi(x) = \langle A_t, \nabla\varphi(x)\rangle - \int_{S^{d-1}} |\langle \frac{\partial}{\partial x}, B\sigma(t,x)\varsigma\rangle|^\alpha \cdot \varphi(x)\mu(d\varsigma).$$

$$\tilde{L}_t^{T,y}\varphi(x) = \langle A_t, \nabla\varphi(x)\rangle - \int_{S^{d-1}} |\langle \frac{\partial}{\partial x}, B\sigma(t,R_{t,T}(y))\varsigma\rangle|^\alpha \cdot \varphi(x)\mu(d\varsigma).$$

Lemma (Control of the kernel)

There exists constants $C > 0,\ \delta > 0$ *s.t. for all* $T \in (0, T_0]$ *and* $(t, x, y) \in [0, T) \times (\mathbb{R}^{nd})^2$*:*

$$|H(t, T, x, y)| \leq C \frac{\delta \wedge |x - R_{t,T}(y)|^{\eta(\alpha \wedge 1)}}{T - t} \tilde{p}_\alpha^{T,y}(t, T, x, y).$$

and the following bound holds:

$$\int_{\mathbb{R}^{nd}} |H(\tau, T, z, y)| dz \leq C(T - \tau)^{\eta(\frac{1}{\alpha} \wedge 1) - 1}.$$

- The last bound reflects the smoothing property (in time) of H.
- Singularity in $\frac{1}{T-t}$, same as in the Brownian setting.
- This is an important ingredient in the proof of the weak uniqueness.
- This holds for any dimension d and any number of oscillators n.

Uniqueness to the Martingale Problem

Suppose we are given two solutions $\mathbb{P}^1$ and $\mathbb{P}^2$ of the martingale problem associated to $(L_s)_{s\in[t,T]}$, starting in x at time t.
Define for a bounded Borel function $f : [0, T] \times \mathbb{R}^{nd} \to \mathbb{R}$,

$$S^i f = \mathbb{E}^i \left(\int_t^T f(s, X_s) ds \right), \ i \in \{1, 2\},$$

where $(X_s)_{s\in[t,T]}$ stands for the canonical process associated with $(\mathbb{P}^i)_{i\in\{1,2\}}$.
We denote:

$$S^\Delta f = S^1 f - S^2 f.$$

We show that $S^\Delta f = 0$ for all $f : [0, T] \times \mathbb{R}^{nd} \to \mathbb{R}$ measurable and bounded, by proving:

$$\|S^\Delta\| := \sup_{|f|_\infty \le 1} |S^\Delta f| = 0.$$

Exploiting the fact that $(\mathbb{P}^i)_{i\in\{1,2\}}$ are solutions to the martingale problem, we have $\forall f \in \mathcal{C}_0^{1,1}([0,T)\times\mathbb{R}^{nd},\mathbb{R})$:

$$f(t,x)+\mathbb{E}^i\left(\int_t^T(\partial_s+L_s)f(s,X_s)ds\right)=0,\ i\in\{1,2\}.$$

Thus, for all $f \in \mathcal{C}_0^{1,1}([0,T)\times\mathbb{R}^{nd},\mathbb{R})$,

$$S^\Delta\Big((\partial_\cdot+L_\cdot)f\Big)=0.$$

Inserting the frozen generator:

$$\underbrace{S^\Delta\Big((\partial_\cdot+\tilde{L}_\cdot)f\Big)}_{\text{Estimates on } \tilde{p}_\alpha}+\underbrace{S^\Delta\Big((L_\cdot-\tilde{L}_\cdot)f\Big)}_{\text{Estimates on } H}=0$$

We take as f the following function:

$$\Psi_\varepsilon(t,x)=\int_{\mathbb{R}^{nd}}dy\int_t^T ds\,\tilde{p}_\alpha^{s+\varepsilon,y}(t,s+\varepsilon,x,y)h(s,y),$$

where h is a, arbitrary test function.

Applying $\partial_t + L_t$ and introducing the frozen generator $\tilde{L}_t$:

$$\begin{aligned}(\partial_t + L_t)\Psi_\varepsilon(t,x) &= (\partial_t + \tilde{L}_t)\Psi_\varepsilon(t,x) + (L_t - \tilde{L}_t)\Psi_\varepsilon(t,x) \\ &= I_1^\varepsilon + I_2^\varepsilon.\end{aligned}$$

We can show that:

$$I_1^\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} -h(t,x), \quad \text{and} \quad |I_2^\varepsilon| \le 1/2|h|_\infty.$$

Also, recall that $S^\Delta\Big((\partial_\cdot + L_\cdot)\Psi_\varepsilon\Big) = 0$ so that $|S^\Delta(I_1^\varepsilon)| = |S^\Delta(I_2^\varepsilon)|$.
We deduce:

$$|S^\Delta h| = \lim_{\varepsilon\to 0}|S^\Delta I_1^\varepsilon| = \lim_{\varepsilon\to 0}|S^\Delta I_2^\varepsilon| \le \|S^\Delta\| \limsup_{\varepsilon\to 0}|I_2^\varepsilon| \le 1/2\|S^\Delta\||h|_\infty.$$

Hence, $\|S^\Delta\| \le 1/2\|S^\Delta\|$. But since $\|S^\Delta\| \le T - t$, we deduce that $\|S^\Delta\| = 0$.

R.F. Bass, and E. Perkins. *A new technique for proving uniqueness for martingale problems.* Asterisque 327 (2009): 47.

F. Delarue, S. Menozzi, *Density estimates for a random noise propagating through a chain of differential equations*, J. Funct. Anal. 259, (2010).

A. Friedman. *Partial differential equation of parabolic type.* Prentice-Hall, (1964).

V. Kolokoltsov, *Symmetric stable laws and stable like diffusions*, London mathematical society (2000).

S. Menozzi, *Parametrix techniques and martingale problems for some degenerate Kolmogorov equations.* Electronic Communications in Probability 16 (2011): 234-250.