Optional semimartingale decomposition and no arbitrage condition in enlarged filtration

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Non-arbitrage up to Random Horizon for Semimartingale Models
T. CHOULLI, A. A., J. DENG and M. JEANBLANC, 2013,
http://arxiv.org/abs/1310.1142

Optional semimartingale decomposition and NUPBR condition in enlarged filtration A. A., T. CHOULLI, and M. JEANBLANC, 2014, Working paper

Problem

- $(\Omega, \mathcal{A}, \mathbb{H}, \mathbb{P})$ is filtered probability space where filtration $\mathbb{H} = (\mathcal{H}_t)_{t\geq 0}$ satisfies usual conditions.
- $X = (X_t)_{t\geq 0}$ is a **price process** of a risky asset, i.e., an $\mathbb{H}$-semimartingale.
- $\theta = (\theta_t)_{t\geq 0}$ is an $\mathbb{H}$-**trading strategy**, i.e., an $\mathbb{H}$-predictable process, integrable w.r.t. X in $\mathbb{H}$.
  By $L^{\mathbb{H}}(X)$ we denote the set of all $\mathbb{H}$-trading strategies.
- $\theta \centerdot X = (\int_0^t \theta_s dX_s)_{t\geq 0}$ is a **wealth process** of $\mathbb{H}$-trading strategy θ.
- Consider $\mathbb{H} \in \{\mathbb{F}, \mathbb{G}\}$ with $\mathbb{F} \subset \mathbb{G}$, i.e., for each $t \geq 0$, $\mathcal{F}_t \subset \mathcal{G}_t$.
  $\mathbb{F}$ represents **regular agent** and $\mathbb{G}$ represents **informed agent**.
- **Assume that there are no arbitrage opportunities in $\mathbb{F}$. Are there arbitrage opportunities in $\mathbb{G}$?**

Non-arbitrage condition – NUPBR

- Let X be an $\mathbb{H}$-semimartingale. We say that X satisfies **No Unbounded Profit with Bounded Risk** (NUPBR($\mathbb{H}$)) if for each $T < \infty$, the set

$$\mathcal{K}_T^{\mathbb{H}}(X) := \left\{(\theta \centerdot X)_T : \theta \in L^{\mathbb{H}}(X) \text{ and } \theta \centerdot X \geq -1\right\}$$

  is bounded in probability.
- The $\mathbb{H}$-semimartingale X satisfies NUPBR($\mathbb{H}$) if there exists $\mathbb{H}$-**local martingale deflator** for X, i.e., a strictly positive $\mathbb{H}$-local martingale L such that LX is an $\mathbb{H}$-local martingale.

Enlargement of filtration

- We say that (H') **hypothesis** is satisfied for $\mathbb{F} \subset \mathbb{G}$ if every $\mathbb{F}$-martingale remains $\mathbb{G}$-semimartingale.
  In such a case we are interested in $\mathbb{G}$-semimartigale decomposition of $\mathbb{F}$-martingale.
- **Progressive enlargement**: Jeulin-Yor's result up to random time.
- **Initial enlargement**: Jacod's results on enlargement of filtration and Stricker-Yor's results on calculus with parameter.

Random times

Let τ be a random time, i.e., a positive random variable.

Consider two $\mathbb{F}$-supermartingales associated to τ:

- The process Z_t defined as $Z_t = \mathbb{P}(\tau > t | \mathcal{F}_t)$
- The process $\widetilde{Z}_t$ defined as $\widetilde{Z}_t = \mathbb{P}(\tau \geq t | \mathcal{F}_t)$

Let A^o be a $\mathbb{F}$-dual optional projection of the process $A = \mathbb{1}_{[\![\tau,\infty[\![}$, i.e., for each optional process Y, A^o satisfies

$$\mathbb{E}(Y_\tau \mathbb{1}_{\{\tau<\infty\}}) = \mathbb{E}(\int_{[0,\infty[} Y_s dA^o_s)$$

Denote by m an $\mathbb{F}$-martingale defined as $m_t = \mathbb{E}(A^o_\infty | \mathcal{F}_t)$. Then

- $Z = m - A^o$ and $\widetilde{Z} = m - A^o_-$
- $\Delta m = \widetilde{Z} - Z_-$ and $\Delta A^o = \widetilde{Z} - Z$

Progressive enlargement of filtration

Progressively enlarged filtration $\mathbb{G}$ associated with τ is defined as

$$\mathcal{G}_t = \bigcap_{s>t} (\mathcal{F}_s \vee \sigma(\tau \wedge s)).$$

Jeulin-Yor's decomposition

For each $\mathbb{F}$-local martingale X, the stopped process X^τ is $\mathbb{G}$-semimartingale with semimartingale decomposition

$$X_t^\tau = \widehat{X}_t + \int_0^{t\wedge\tau} \frac{1}{Z_{s-}} d\langle X, m\rangle_s^{\mathbb{F}}$$

where $\widehat{X}$ is $\mathbb{G}$-local martingale.

$\mathbb{F}$-stopping time R

- The three sets $\{\widetilde{Z} = 0\}$, $\{Z = 0\}$ and $\{Z_- = 0\}$ have the same début which is an $\mathbb{F}$-stopping time

$$R := \inf\{t \geq 0 : Z_t = 0\}$$

- We decompose R as $R = \widetilde{R} \wedge \widehat{R} \wedge \bar{R}$ with $\mathbb{F}$-stopping times:

$$\widetilde{R} := R_{\{\widetilde{Z}_R = 0 < Z_{R-}\}} \quad \widehat{R} := R_{\{\widetilde{Z}_R > 0\}} \quad \text{and} \quad \bar{R} := R_{\{Z_{R-} = 0\}}$$

- Notice that $\widetilde{Z}_\tau > 0$ and $Z_{\tau-} > 0$.

Optional semimartingale decomposition in $\mathbb{G}$

For $\mathbb{H}$-locally integrable variation process V, we denote by $V^{p,\mathbb{H}}$ its $\mathbb{H}$-dual predictable projection, i.e., an $\mathbb{H}$-predictable finite variation process such that for each $\mathbb{H}$-predictable process Y, $V^{p,\mathbb{H}}$ satisfies

$$\mathbb{E}(\int_{[0,\infty[} Y_s dV_s) = \mathbb{E}(\int_{[0,\infty[} Y_s dV_s^{p,\mathbb{H}}).$$

Theorem

Let X be a $\mathbb{F}$-local martingale. Then

$$X_t^{\tau} = \bar{X}_t + \int_0^{t\wedge\tau} \frac{1}{\widetilde{\widetilde{Z}}_s} d[X,m]_s - \left(\Delta X_{\widetilde{R}} \mathbb{1}_{[\![\widetilde{R},\infty[\![}\right)^{p,\mathbb{F}}_{t\wedge\tau}$$

where $\bar{X}$ is a $\mathbb{G}$-local martingale.

Projections in $\mathbb{G}$ in terms of projections in $\mathbb{F}$

Lemma

Let V be an $\mathbb{F}$-adapted process with locally integrable variation. Then, we have

- $(V^\tau)^{p,\mathbb{G}} = \frac{1}{Z_-} 1\!\!1_{[\![0,\tau]\!]} \centerdot (\widetilde{Z} \centerdot V)^{p,\mathbb{F}}$.
- *The process*

$$U := \frac{1}{\widetilde{Z}} 1\!\!1_{[\![0,\tau]\!]} \centerdot V$$

  is locally integrable variation process in $\mathbb{G}$ and

$$U^{p,\mathbb{G}} = \frac{1}{Z_-} 1\!\!1_{[\![0,\tau]\!]} \centerdot (1\!\!1_{\{\widetilde{Z}>0\}} \centerdot V)^{p,\mathbb{F}}.$$

$\mathbb{G}$-local martingale deflator

- Defined $\mathbb{G}$-local martingale

$$\bar{N} = \frac{1}{Z_-} \mathbb{1}_{[\![0,\tau]\!]} \centerdot \bar{m}$$

- Then, continuous martingale part and jump process of $\bar{N}$ are of the form

$$\bar{N}^c = \frac{1}{Z_-} \mathbb{1}_{[\![0,\tau]\!]} \centerdot \bar{m}^c$$

$$\Delta \bar{N} = \frac{1}{Z_-} \Delta \bar{m} \mathbb{1}_{[\![0,\tau]\!]} = \frac{\Delta m}{\widetilde{Z}} \mathbb{1}_{[\![0,\tau]\!]} - {}^{p,\mathbb{F}} \left(\mathbb{1}_{[\![\widetilde{R}]\!]} \right) \mathbb{1}_{[\![0,\tau]\!]}$$

- Clearly

$$-\Delta \bar{N} \geq \left(-1 + \frac{Z_-}{\widetilde{Z}} \right) \mathbb{1}_{[\![0,\tau]\!]} > -1$$

$\mathbb{G}$-local martingale deflator

Theorem

Let $L = \mathcal{E}\left(-\bar{N}\right)$. Then, for any $\mathbb{F}$-local martingale X, the process

$$LX^{\tau} - L_{-} \cdot \left(\left(\Delta X_{\widetilde{R}} + \frac{\Delta \langle m, X \rangle_{\widetilde{R}}}{Z_{\widetilde{R}-}} \right) \mathbb{1}_{[\![\widetilde{R},\infty[\![} \right)^{p,\mathbb{F}}_{\cdot \wedge \tau}$$

is a $\mathbb{G}$-local martingale.

Corollary

If X is quasi-left continuous and $\Delta X_{\widetilde{R}} = 0$ on $\{\widetilde{R} < \infty\}$, then L is a $\mathbb{G}$-local martingale deflator for X^{τ}.

Non-arbitrage up to Random Horizon

Theorem

Let τ be a random time. Then, the following are equivalent:

1. *The thin set* $\{\widetilde{Z} = 0 \ \& \ Z_- > 0\}$ *is evanescent.*
2. $\mathbb{F}$*-stopping time* $\widetilde{R} = \infty$*.*
3. *For any* $\mathbb{F}$*-local martingale* X*, process* $X^\tau L$ *is a* $\mathbb{G}$*-local martingale.*
4. *For any process* X *satisfying* $NUPBR(\mathbb{F})$*,* X^τ *satisfies* $NUPBR(\mathbb{G})$*.*

Proof

- 2. $\Rightarrow$ 3. Optional decomposition
- 3. $\Rightarrow$ 2. $\mathbb{F}$-martingale $X = \mathbb{1}_{[\![\widetilde{R},\infty[\![} - \left(\mathbb{1}_{[\![\widetilde{R},\infty[\![}\right)^{p,\mathbb{F}}$ stopped at τ:

$$X^{\tau} = -\left(\mathbb{1}_{[\![\widetilde{R},\infty[\![}\right)^{p,\mathbb{F}}_{\cdot\wedge\tau}$$

  yields that $\widetilde{R} = \infty$
- 3. $\Rightarrow$ 4. Takaoka's characterization, localization and equivalent change of measure

Initial enlargement of filtration

- Initially enlarged filtration $\mathbb{G}$ associated with random variable ξ is defined as

$$\mathcal{G}_t = \bigcap_{s>t} (\mathcal{F}_s \vee \sigma(\xi)).$$

- **Jacod's hypothesis**

  A real-valued random variable ξ satisfies Jacod's hypothesis if there exists a σ-finite positive measure η such that for every $t \geq 0$

  $\mathbb{P}(\xi \in du | \mathcal{F}_t)(\omega) \ll \eta(du)$ $\mathbb{P}$-a.s.

- As shown by Jacod, without loss of generality, η can be taken as law of ξ in the above definition.

Parameterized processes

- Stricker-Yor's calculus with parameter
- Consider a mapping $X : (t, \omega, u) \to X_t^u(\omega)$ with values in $\mathbb{R}$ on $\mathbb{R}_+ \times \Omega \times \mathbb{R}$.
- Let $\mathcal{J}$ be a class of $\mathbb{F}$-optional processes, for example the class of $\mathbb{F}$-(local) martingales or the class of $\mathbb{F}$-locally integrable variation processes.
  Then, $(X^u, u \in \mathbb{R})$ is called a parametrized $\mathcal{J}$-process if for each $u \in \mathbb{U}$ the process X^u belongs to $\mathcal{J}$ and if it is measurable with respect to $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R})$.

Initial enlargement under Jacod's hypothesis

- For ξ satisfying Jacod's hypothesis, there exists a parameterized positive $\mathbb{F}$-martingale $(q^u, u \in \mathbb{R})$ such that for every $t \geq 0$, the measure $q_t^u(\omega)\eta(du)$ is a version of $\mathbb{P}(\xi \in du|\mathcal{F}_t)(\omega)$.
- **Semimartingale decomposition**

  Let $(X^u, u \in \mathbb{R})$ be a parameterized $\mathbb{F}$-local martingale. Then

$$X_t^\xi = \widehat{X}_t^\xi + \int_0^t \frac{1}{q_{s-}^\xi} d\langle X^u, q^u\rangle_s^{\mathbb{F}}|_{u=\xi}$$

  where $\widehat{X}^\xi$ is an $\mathbb{G}$-local martingale.

$\mathbb{F}$-stopping times R^u

- For each u define $\mathbb{F}$-stopping time

$$R^u = \inf\{t : q_t^u = 0\}.$$

- We have $q^u > 0$ and $q_-^u > 0$ on $[\![0, R^u[\![$ and $q^u = 0$ on $[\![R^u, \infty[\![$.
- $\mathbb{G}$-stopping time $R^\xi = \infty$ a.s. or equivalently $q_t^\xi > 0$ and $q_{t-}^\xi > 0$ for $t \geq 0$ $\mathbb{P}$-a.s. Let us decompose $\mathbb{F}$-stopping time R^u as $R^u = \widetilde{R}^u \wedge \bar{R}^u$ with

$$\widetilde{R}^u = R^u_{\{q^u_{R^u-} > 0\}} \quad \text{and} \quad \bar{R}^u = R^u_{\{q^u_{R^u-} = 0\}}.$$

Optional semimartingale deomposition in $\mathbb{G}$

Theorem

Let $(X^u, u \in \mathbb{R})$ be a parameterized $\mathbb{F}$-local martingale. Then X^ξ decomposes as $\mathbb{G}$-semimartingale as

$$X_t^\xi = \bar{X}_t^\xi + \int_0^t \frac{1}{q_s^\xi} d[X^\xi, q^\xi]_s - \left(\Delta X^u_{\widetilde{R}^u} \mathbb{1}_{[\![\widetilde{R}^u, \infty[\![} \right)^{p,\mathbb{F}} |_{u=\xi}$$

where $\bar{X}^\xi$ is a $\mathbb{G}$-local martingale.

Projections in $\mathbb{G}$ in terms of projections in $\mathbb{F}$

Lemma

Let $(V^u, u \in \mathbb{R})$ be a parameterized $\mathbb{F}$-adapted càdlàg process with locally integrable variation. Then,

- *The $\mathbb{G}$-dual predictable projection of V^ξ is*

$$\left(V^\xi\right)^{p,\mathbb{G}} = \frac{1}{q^\xi_-} \centerdot \left(q^u \centerdot V^u\right)^{p,\mathbb{F}}|_{u=\xi}.$$

- *If V belongs to $\mathcal{A}^+_{loc}(\mathbb{F})$, then the process $U := \frac{1}{q^\xi} \centerdot V$ belongs to $\mathcal{A}^+_{loc}(\mathbb{G})$.*
- *The parametrized process $(U^u, u \in \mathbb{R})$ is well defined, its variation is $\mathbb{G}$-locally integrable, and $\mathbb{G}$-dual predictable projection of U^ξ is given by*

$$(U^\xi)^{p,\mathbb{G}} = \frac{1}{q^\xi_-} \centerdot \left(\mathbb{1}_{\{q^u>0\}} \centerdot V^u\right)^{p,\mathbb{F}}|_{u=\xi}.$$

$\mathbb{G}$-local martingale $\bar{q}^{\xi}$

Lemma

Take the following $\mathbb{G}$-local martingale

$$\bar{q}^{\xi} := q^{\xi} - \frac{1}{q^{\xi}} \centerdot [q^{\xi}] - q^{\xi}_{-} \centerdot \left(\mathbb{1}_{[\![\widetilde{R}^{u},\infty[\![}\right)^{p,\mathbb{F}} |_{u=\xi}.$$

- *The $\mathbb{G}$-predictable process $\frac{1}{q^{\xi}_{-}}$ is integrable with respect to $\bar{q}^{\xi}$.*
- *The $\mathbb{G}$-local martingale*

$$N := \frac{1}{q^{\xi}_{-}} \centerdot \bar{q}^{\xi}$$

has continuous martingale part and jump equal respectively to

$$N^{c} = \frac{1}{q^{\xi}_{-}} \centerdot \left((q^{\xi})^{c} - \frac{1}{q^{\xi}_{-}} \centerdot \langle (q^{\xi})^{c} \rangle^{\mathbb{F}}\right) \quad \Delta N = \frac{\Delta q^{\xi}}{q^{\xi}} - {}^{p,\mathbb{F}}\left(\mathbb{1}_{[\![\widetilde{R}^{u}]\!]}\right) |_{u=\xi}.$$

$\mathbb{G}$-supermartingale $\frac{1}{q^\xi}$

Lemma

- *The process* $\frac{1}{q^\xi}$ *is* $\mathbb{G}$*-supermartingale with decomposition*

$$\frac{1}{q^\xi} = 1 - \frac{1}{(q^\xi_-)^2} \cdot \bar{q}^\xi - \frac{1}{q^\xi_-} \cdot \left(\mathbb{1}_{[\![\widetilde{R}^u, \infty[\![} \right)^{p,\mathbb{F}} |_{u=\xi}.$$

  Moreover, it can be written as stochastic exponential of the form

$$\frac{1}{q^\xi} = \mathcal{E}\left(-\frac{1}{q^\xi_-} \cdot \bar{q}^\xi - \left(\mathbb{1}_{[\![\widetilde{R}^u, \infty[\![} \right)^{p,\mathbb{F}} |_{u=\xi} \right).$$

- *The process* $\frac{1}{q^\xi}$ *is* $\mathbb{G}$*-local martingale if and only if* $\widetilde{R}^u = \infty$ $\mathbb{P} \otimes \eta$*-a.s.* *Then* $\frac{1}{q^\xi} = \mathcal{E}\left(-\frac{1}{q^\xi_-} \cdot \bar{q}^\xi \right) = \mathcal{E}\left(-N\right)$.

$\mathbb{G}$-local martingale deflator

Theorem

Let $L = \mathcal{E}(-N)$. Then, for any parametrized $\mathbb{F}$-local martingale $(X^u, u \in \mathbb{R})$, the process

$$LX^{\xi} - L_{-} \centerdot \left(\left(\Delta X^u_{\widetilde{R}^u} + \frac{\Delta \langle q^u, X \rangle_{\widetilde{R}^u}}{q^u_{\widetilde{R}^u -}} \right) \mathbb{1}_{[\![\widetilde{R}^u, \infty [\![} \right)^{p, \mathbb{F}} \Big|_{u=\xi}$$

is a $\mathbb{G}$-local martingale.

Corollary

If X is quasi-left continuous and $\Delta X^u_{\widetilde{R}^u} = 0$ on $\{R^u < \infty\}$ $\mathbb{P} \otimes \eta$-a.s., then L is a $\mathbb{G}$-local martingale deflator for X^{ξ} in $\mathbb{G}$.

NUPBR condition for initial enlargement under Jacod's hypothesis

Theorem

The following conditions are equivalent:

1. *The thin set* $\{q^u = 0 < q^u_-\}$ *is evanescent* η*-a.a.*
2. *The* $\mathbb{F}$*-stopping time* $\widetilde{R}^u = \infty$ $\mathbb{P} \otimes \eta$*-a.s.*
3. *If* $(X^u, u \in \mathbb{R})$ *is parameterized* $\mathbb{F}$*-local martingale, then* X^ξ *satisfies NUPBR(*$\mathbb{G}$*). Morover,* $\frac{X^\xi}{q^\xi}$ *is a* $\mathbb{G}$*-local martingale, i.e.,* $\frac{1}{q^\xi}$ *is its* $\mathbb{G}$*-local martingale deflator.*

Proof

- 2. $\Rightarrow$ 3. Optional decomposition under Jacod's hypothesis
- 3. $\Rightarrow$ 2. Parameterized $\mathbb{F}$-martingale $(X^u, u \in \mathbb{R})$ with $X^u = \mathbb{1}_{[\![\widetilde{R}^u,\infty[\![} - \left(\mathbb{1}_{[\![\widetilde{R}^u,\infty[\![}\right)^{p,\mathbb{F}}$ "at ξ":

$$X^\xi = -\left(\mathbb{1}_{[\![\widetilde{R}^u,\infty[\![}\right)^{p,\mathbb{F}}|_{u=\xi}$$

  yields that $\widetilde{R}^u = \infty$ $\mathbb{P} \otimes \eta$-a.s.

Optional semimartingale decomposition in $\mathbb{G}$

- Let $\mathbb{Q}$ be a probability absolutely continuous with respect to $\mathbb{P}$, and let $\zeta_t = \mathbb{E}_{\mathbb{P}}(\frac{d\mathbb{Q}}{d\mathbb{P}}|\mathcal{F}_t)$, $S = \inf\{t > 0 : \zeta_t = 0\}$ and $\widetilde{S} = S_{\{\zeta_{S-}>0\}}$.
  Let X be a $\mathbb{P}$-local martingale. Then, X has decomposition under $\mathbb{Q}$:

$$X = \bar{X} + \frac{1}{\zeta} \centerdot [X, \zeta] - \left(\Delta X_{\widetilde{S}} \mathbb{1}_{[\![\widetilde{S},\infty[\![}\right)^{p,\mathbb{P}}$$

  where $\bar{X}$ is $\mathbb{Q}$-local martingale.

- Up to random time τ:

$$X^\tau = \bar{X} + \frac{1}{\widetilde{Z}_s} \mathbb{1}_{[\![0,\tau]\!]} \centerdot [X, m] - \left(\Delta X_{\widetilde{R}} \mathbb{1}_{[\![\widetilde{R},\infty[\![}\right)^{p,\mathbb{F}}_{\cdot\wedge\tau}$$

- Under Jacod's hypothesis:

$$X^\xi = \bar{X}^\xi + \frac{1}{q^\xi} \centerdot [X, q^\xi] - \left(\Delta X^u_{\widetilde{R}^u} \mathbb{1}_{[\![\widetilde{R}^u,\infty[\![}\right)^{p,\mathbb{F}} |_{u=\xi}$$

Thank you!