REGULARITY TEAM (INRIA Saclay / Ecole Centrale Paris).

Stochastic calculus with respect to the Rosenblatt process

Onzième Colloque: Jeunes Probabilistes et Statisticiens

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Motivation

Why ?

- Fractional Brownian motion (fBm) is not a semimartingale for $H \neq \frac{1}{2}$: interesting theoretical problem.
- Popular model in diverse applications: hydrology, telecommunications, fluid dynamics, mathematical finance.
- Rosenblatt process: simplest **non-Gaussian** Hermite processes.

Motivation

20 years of research

- Pathwise methods: Lin (1995), Zhäle (1998), Coutin and Qian (2002), Gradinaru, Nourdin, Russo and Vallois (2005)...
- Malliavin calculus: Deucreusefond and Üstünel (1999), Alos, Nualart and Mazet (2001), Cheredito and Nualart (2005)...
- White noise distribution theory: Elliott and Van Der Hoek (2003), Bender (2003), Hu and Oksendal (2003)...
- Isometric construction: Mishura and Valkeila (2000).
- Approximation: Carmona, Coutin and Montseny (2001).

Fractional Brownian motion

Kolmogorov (1940)

Fractional Brownian motion $\{B_t^H\}$ is the unique centered gaussian process whose covariance function is equal to:

$$\mathbb{E}[B_t^H B_s^H] = \frac{1}{2}[|t|^{2H} + |s|^{2H} - |t-s|^{2H}].$$

Mandelbrot (1968)

Let $0 < H < 1$ and $\{B_x\}_{x\in\mathbb{R}}$ be a Brownian motion.

$$B_t^H = \int_{\mathbb{R}} \left[(t-x)_+^{H-\frac{1}{2}} - (-x)_+^{H-\frac{1}{2}} \right] dB_x.$$

Multiple Wiener-Itô Integrals

Definition

The multiple Wiener Itô integral is a continuous linear application from $\tilde{L}^2(\mathbb{R}^d)$ to $L^2(\Omega, \mathscr{F}, \mathbb{P})$, where $\tilde{L}^2(\mathbb{R}^d)$ is the space of square-integrable symmetric functions.

Properties

- $I_d(f) = I_d(\tilde{f})$ where $\tilde{f} = \dfrac{1}{d!} \sum_{\sigma \in \mathscr{S}_d} f \circ \sigma$.
- $\mathbb{E}[I_p(f) I_q(g)] = p! < \tilde{f}, \tilde{g} > \delta_{p,q}$.
- $\mathbb{E}[(I_d(f))^2] = d! ||\tilde{f}||_2^2$

Dobrushin, Major (1979) and Taqqu (1979)

Hermite processes

Let $H \in (\frac{1}{2}, 1)$. Let $\{\xi_n; n \in \mathbb{Z}\}$ be a Gaussian stationary sequence with mean zero, unit variance and $\mathbb{E}[\xi_0 \xi_n] \equiv n^{\frac{2H-2}{d}} L(n)$. Let g be a function such that $\mathbb{E}[g(\xi_0)] = 0$, $\mathbb{E}[g(\xi_0)^2] < \infty$ and d as Hermite rank. Then:

$$\forall (t_1, \dots, t_p) \in \mathbb{R}_+^p \quad \left(\frac{1}{n^H} \sum_{i=1}^{\lfloor nt_1 \rfloor} g(\xi_i), \dots, \frac{1}{n^H} \sum_{i=1}^{\lfloor nt_p \rfloor} g(\xi_i) \right) \Rightarrow \left(I_d(h_{t_1}^{H,d}), \dots, I_d(h_{t_p}^{H,d}) \right)$$

where $h_t^{H,d} = \int_0^t \prod_{j=1}^d (s - x_j)_+^{-(\frac{1}{2} + \frac{1-H}{d})} ds$

Definition

Representation

$$\forall t \in \mathbb{R}_+ \quad X_t^H = c(H) \int_{\mathbb{R}^2} \left(\int_0^t (s - x_1)_+^{\frac{H}{2}-1} (s - x_2)_+^{\frac{H}{2}-1} ds \right) dB_{x_1} dB_{x_2}$$

where $H \in (\frac{1}{2}, 1)$ and $c(H)$ is a normalizing constant such that $\mathbb{E}[|X_1^H|^2] = 1$.

Properties

- Non-Gaussian process.
- Same covariance function as fBm $\Rightarrow$ Long-range dependency.
- $H - \delta$, $\delta > 0$,Hölder continuous.
- Not a semimartingale.
- Zero quadratic variation.

Stochastic calculus with respect to the Rosenblatt process: Russo-Vallois regularization

Tudor 2008

- Forward integtal of $Y.$ (continuous) with respect to $X.^H$:

$$\int_0^T Y_t d^+ X_t^H = \lim_{\epsilon \to 0^+} - ucp \quad \int_0^T Y_t \frac{X_{t+\epsilon}^H - X_t^H}{\epsilon} dt.$$

- For $f \in C^2(\mathbb{R})$,

$$f(X_t^H) - f(X_0^H) = \int_0^t f'(X_s^H) dX_s^H.$$

Stochastic calculus with respect to the Rosenblatt process: Skorohod type integral

Tudor 2008

- $\forall t \in [0,T] \quad Z_t^H = c(H) \int_{[0;t]^2} \int_0^t \prod_{j=1}^2 (\frac{s}{x_j})^{\frac{H}{2}} (s - x_j)_+^{\frac{H}{2}-1} ds dB_{x_1} dB_{x_2}$
- Let $\{Y_t : t \in [0;T]\}$ be a square integrable stochastic process.

$$\int_0^T Y_t \delta Z_t^H = \delta^2(I_H(Y)).$$

- If $\{Y_t\}$ sufficiently regular (in the Malliavin sense): upper bound for the variance of $\int_0^T Y_t \delta Z_t^H$.
- Similarly: $\{\int_0^t Y_s \delta Z_s^H\}$ is $H - \delta$ Hölder continuous.

Stochastic calculus with respect to the Rosenblatt process

Itô formula in the divergence sense

For $f \in C^2(\mathbb{R})$,

$$f(Z_t^H) - f(Z_0^H) = \int_0^T f'(Z_t^H)\delta Z_t^H + 2Tr^{(1)}(D^{(1)}f'(Z_t^H)) - Tr^{(2)}(D^{(2)}f'(Z_t^H)).$$

if the trace terms exist.

Remark, (Tudor 2008)

- For $f = x^3$, appearance of a term invovling f'''.
- Non-zero cumulants of the Rosenblatt distribution (law of Z_1^H) appear for $f = x^2$ and for $f = x^3$
- What does Itô's formula look like for general f smooth enough ?

Hida distributions

Setting

- $(\Omega, \mathscr{F}, \mathbb{P}) = (S'(\mathbb{R}), \mathscr{F}_*, \mu)$.
- $\mu - a.e. \quad \forall t \geq 0 \quad B_t(.) =< .; 1_{[0;t]} >$.
- $<; f >= \int_{\mathbb{R}} f(s) dB_s$.
- $(L^2) = L^2(\Omega, \mathscr{F}, \mathbb{P})$.
- $(S) \subset (L^2) \subset (S)^*$.

(S)-transform

Let $\Phi \in (S)^*$. For every function $\xi \in S(\mathbb{R})$, we define the S-transform of Φ by:

$$S(\Phi)(\xi) = \langle\langle \Phi; : \exp(<; \xi >) : \rangle\rangle$$

where $: \exp(<; \xi >) := \exp(<; \xi > - \frac{||\xi||^2_{L^2(\mathbb{R})}}{2}) = \sum_{n=0}^{\infty} \frac{I_n(\xi^{\otimes n})}{n!} \in (S)$.

Hida dristributions

Definitions

- $\Phi \diamond \Psi \in (S)^*$ defined by:
  $\forall \xi \in S(\mathbb{R}), \quad S(\Psi)(\xi)S(\Phi)(\xi) = S(\Phi \diamond \Psi)(\xi).$
- $\forall y \in S'(\mathbb{R})$, D_y linear continuous operator from (S) to (S) such that:
  $$D_y(I_n(\phi_n)) = nI_{n-1}(y \otimes_1 \phi_n).$$
- $\forall y \in S'(\mathbb{R}), \forall \Psi \in (S)^*$, D_y^* linear continuous operator from $(S)^*$ into itself such that:
  $$\forall \xi \in S(\mathbb{R}) \quad S(D_y^*(\Psi))(\xi) = < y; \xi > S(\Psi)(\xi) = S(I_1(y) \diamond \Psi)(\xi)$$

$(S)^*$-differentiability and $(S)^*$-integrability

$(S)^*$-derivatives

- White noise: $\dot{B}_t = I_1(\delta_t)$.
- Fractional noise: $\dot{B}_t^H = I_1(\delta_t \circ (I_+^{H-\frac{1}{2}}))$, where $I_+^{H-\frac{1}{2}}$ fractional integral of order $H - \frac{1}{2}$.

$(S)^*$-integrability

$Y : I \longrightarrow (S)^*$ is integrable if:

- $\forall \xi \in S(\mathbb{R})$, $S(Y_.)(\xi)$ is measurable on I.
- $\forall \xi \in S(\mathbb{R})$, $S(Y_.)(\xi) \in L^1(I)$.
- $\int_I S(Y_t)(\xi)dt$ is the S-transform of a certain Hida distribution.

White noise integral and fractional noise integral

Kubo and Takenaka, (1981)

Let $\{Y_t : t \in [0;T]\}$ be a non-anticipating stochastic process in $L^2([0,T] \times \Omega)$. Then,

$$\int_0^T Y_t dB_t = \int_0^T Y_t \diamond \dot{B}_t dt = \int_0^T D^*_{\delta_t}(Y_t) dt.$$

Bender (2003) and Elliott et al. (2003)

Let $\{Y_t; t \in [0;T]\}$ be a $(S)^*$ stochastic process which is $(S)^*$ integrable. The fractional noise integral of $Y_.$ over $[0,T]$ is defined by:

$$\int_0^T Y_t dB^H_t = \int_0^T Y_t \diamond \dot{B}^H_t dt = \int_0^T D^*_{\delta_t \circ I_+^{H-\frac{1}{2}}}(Y_t) dt.$$

Rosenblatt noise integral

Rosenblatt noise

It is defined by:

$$\forall t > 0 \quad \dot{X}_t^H = d(H) I_2(\delta_t^{\otimes 2} \circ (I_+^{\frac{H}{2}})^{\otimes 2})$$

and characterized by:

$$\forall \xi \in S(\mathbb{R}) \quad S(\dot{X}_t^H)(\xi) = d(H)(I_+^{\frac{H}{2}}(\xi)(t))^2$$

Rosenblatt noise integral

Let $\{Y_t; t \in [0;T]\}$ be a $(S)^*$ stochastic process which is $(S)^*$ integrable. The Rosenblatt noise integral of $Y_.$ over $[0,T]$ is defined by:

$$\int_0^T Y_t dX_t^H = \int_0^T Y_t \diamond \dot{X}_t^H dt = \int_0^T (D^*_{\sqrt{d(H)}\delta_t \circ I_+^{\frac{H}{2}}})^2 (Y_t) dt$$

Itô formula

Theorem (Arras 2013)

Let $(a,b) \in \mathbb{R}_+^*$ such that $a \leq b < \infty$. Let F be an entire analytic function of the complex variable verifying:

$$\exists N \in \mathbb{N}, \exists C > 0, \forall z \in \mathbb{C} \quad |F(z)| \leq C(1+|z|)^N \exp\left(\frac{1}{\sqrt{2}b^H}|\Im(z)|\right)$$

Then, in $(S)^*$:

$$F(X_b^H) - F(X_a^H) = \int_a^b F^{(1)}(X_t^H) \diamond \dot{X}_t^H dt$$
$$+ \sum_{k=2}^{\infty} \left(H\kappa_k(X_1^H) \int_a^b \frac{t^{Hk-1}}{(k-1)!} F^{(k)}(X_t^H) dt \right.$$
$$\left. + 2^k \int_a^b F^{(k)}(X_t^H) \diamond \dot{X}_t^{H,k} dt \right)$$

Comments

Remarks

- All the derivatives of F are involved.
- Non-zero cumulants, $\kappa_k(X_1^H)$, appear in the formula.
- Appearance of $\{X_t^{H,k} : t \geq 0\}$ defined by:

$$X_t^{H,k} = \int_{\mathbb{R}} \int_{\mathbb{R}} \underbrace{(\ldots((f_t^H \otimes_1 f_t^H) \otimes_1 f_t^H) \ldots \otimes_1 f_t^H)}_{k-1 \times \otimes_1}(x_1, x_2) dB_{x_1} dB_{x_2}$$

where f_t^H is the kernel of the Rosenblatt process.

Proof

$$S(F(X_t^H))(\xi) = \int_{S'(\mathbb{R})} F((X_t^H)(x))d\mu_\xi(x) = \frac{1}{2\pi} < \mathscr{F}(F); \mathbb{E}^{\mu_\xi}[\exp(i.X_t^H)] > .$$

By Paley-Wiener theorem, $\mathscr{F}(F)$ has compact support contained in $\{\theta : |\theta| \leq \frac{1}{\sqrt{2}b^H}\}$.
Thus, we need to know the behaviour of $\theta \to \mathbb{E}^{\mu_\xi}[\exp(i\theta X_t^H)]$ around the origin.

Proof

Lemma

Let $\xi \in S(\mathbb{R})$ and $t > 0$. We have (for θ being small enough):

$$\begin{aligned}\mathbb{E}^{\mu_\xi}[\exp(i\theta X_t^H)] &= \exp(i\theta < f_t^H; \xi^{\otimes 2} >) \\ &\times \exp\left(\sum_{k=2}^{+\infty} \frac{(i\theta t^H)^k}{k!} \kappa_k(X_1^H)\right) \\ &\times \exp\left(\sum_{k=2}^{+\infty} (2i\theta)^k S(X_t^{H,k})(\xi)\right)\end{aligned}$$

Differentiate $S(F(X_t^H))(\xi)$ with respect to t. Intervene duality bracket and differentiation. Integrate over $[a,b]$.

End

THANK YOU !

Comparison with other approaches

Theorem (Arras 2013)

Let $\{Y_t; t \in [0;T]\}$ be a stochastic process such that $Y \in L^2(\Omega;\mathscr{H}) \cap L^2([0,T];\mathbb{D}^{2,2})$ and $\mathbb{E}[\int_0^T \int_0^T ||D^2_{s_1,s_2} Y||^2_{\mathscr{H}} ds_1 ds_2] < \infty$ where

$$\mathscr{H} = \{f : [0;T] \to \mathbb{R}; \int_0^T \int_0^T f(s)f(t)|t-s|^{2H-2} dsdt < \infty\}.$$

Then, $\{Y_t\}$ is Skorohod integrable and $(S)^*$-integrable with respect to the Rosenblatt process, $\{Z_t^H\}_{t \in [0;T]}$, and we have:

$$\int_0^T Y_t \delta Z_t^H = \int_0^T Y_t \diamond \dot{Z}_t^H dt$$