Région PACA

Independence Tests between Point Processes

Mélisande ALBERT

Independence tests between two point processes. Application to the study of neuronal spike trains synchronization

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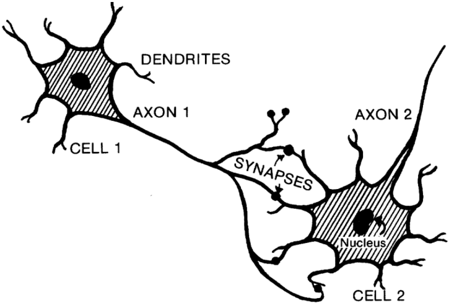
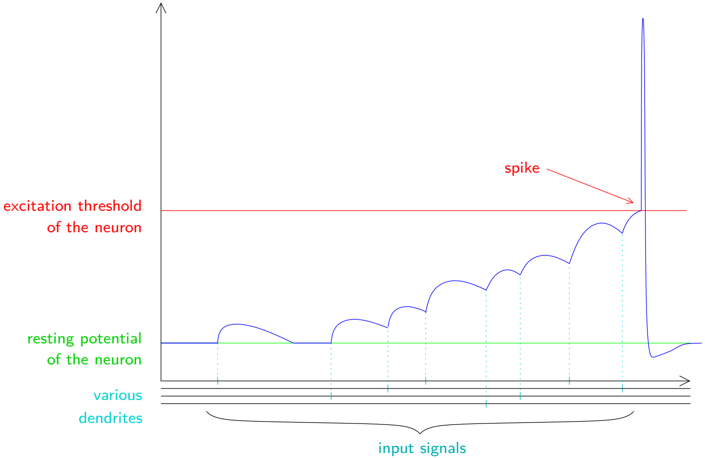
The neuronal information is transmitted by the spikes/action potentials.



Electric potential of the cell
Synaptic Integration



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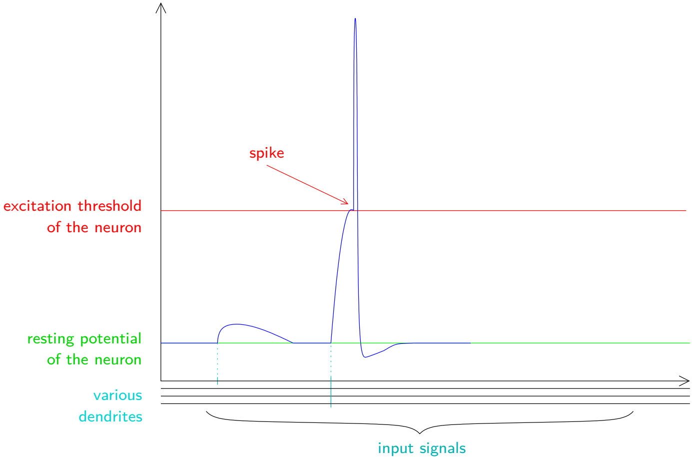


[Grammont and Riehle (1999)]: neurons coordinate their activity at very precise moments.

Problematic

Detection of dependences between neurons.

Statistical Modeling for Neuronal Activity

Spike = "all-or-none" phenomenon,

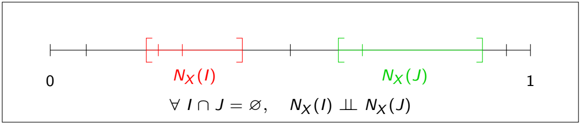
$\Rightarrow$ Model: point processes on a time slot $[0, 1]$.

Definition

Point process on $[0, 1]$ = random countable set of points in $[0, 1]$.

$\mathcal{X}$:= the set of almost surely finite Point process on $[0, 1]$.

Example: X homogeneous Poisson process with intensity $\lambda > 0$.



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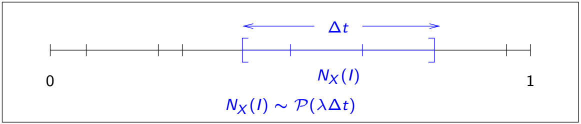

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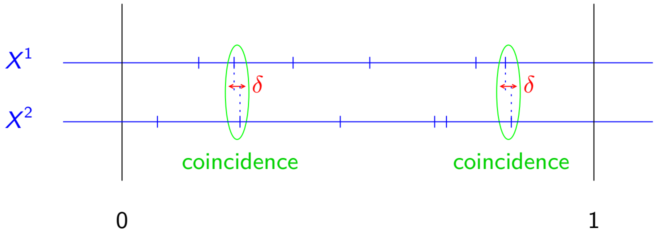
For X a point process in $[0, 1]$, we introduce

$$dN_X = \sum_{T \in X} \delta_T.$$

Notion of (delayed) coincidence [Grün et al. (1999)]

φ_δ^{coinc} counts the number of coincidences between two point processes:

$$\varphi_\delta^{coinc}\left(X^1, X^2\right) = \int_a^b \int_a^b \mathbb{1}_{\{|u-v|\leq\delta\}} dN_{X^1}(u) dN_{X^2}(v).$$



Unitary Events [Grün et al. (1999)]

Most well known method using delayed coincidences.

Problem:

- Very few methods theoretically justified.

Unitary Events [Grün et al. (1999)]

Most well known method using delayed coincidences.

A correction from Tuleau-Malot et al. (2012)

Homogeneous Poisson process assumption $\Rightarrow$ Determination of the asymptotic distribution of the test statistic.

Problems:

- Homogeneous Poisson process assumption questionable,
- Too few data to use the asymptotic distribution.

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Aim

To construct a non-asymptotic independence test between point processes with no assumption on the underlying distribution.

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Pipa et al. (2003)

Trial shuffling for another notion of coincidences.

X^m: activity of neuron $m \in \{1,2\}$ on $[0,1]$.

$$(H_0) : X^1 \perp\!\!\!\perp X^2 \quad \text{against} \quad (H_1) : X^1 \not\perp\!\!\!\perp X^2$$

Often, in $\mathbb{R}^k$, tests based on $\iint \varphi(x^1, x^2) \left(dP_{(x^1,x^2)} - dP^1_{x^1} dP^2_{x^2} \right)$.

For φ well chosen,

$$\text{joint distribution} \leftrightarrow \text{product of marginals}.$$

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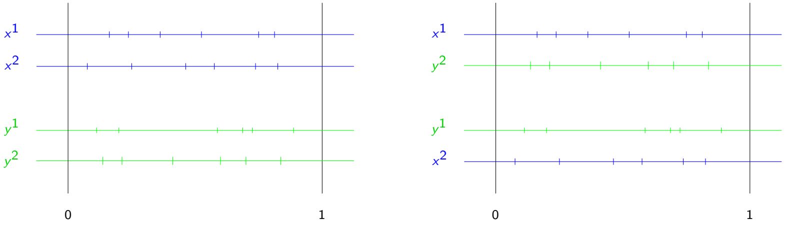
Let $\mathbb{X}_n = (X_i)_{1\leq i\leq n}$, where $X_i = \left(X_i^1, X_i^2\right)$ i.i.d. $\sim P$ in $\mathcal{X}\times\mathcal{X}$.

Unbiaised estimator:

$$\frac{1}{n(n-1)}\sum_{i\neq i'}\left(\varphi\left(X_i^1,X_i^2\right) - \varphi\left(X_i^1,X_{i'}^2\right)\right)$$

Let $h_\varphi\left((x^1,x^2),(y^1,y^2)\right) = \varphi(x^1,x^2) + \varphi(y^1,y^2) - \varphi(x^1,y^2) - \varphi(y^1,x^2)$,



$\varphi(x^1,x^2)+\varphi(y^1,y^2)$ $\qquad$ $-\varphi(x^1,y^2)-\varphi(y^1,x^2)$

X^m: activity of neuron $m \in \{1,2\}$ on $[0,1]$.

$$(H_0) : X^1 \perp\!\!\!\perp X^2 \quad \text{against} \quad (H_1) : X^1 \not\perp\!\!\!\perp X^2$$

Often, in $\mathbb{R}^k$, tests based on $\iint \varphi(x^1, x^2) \left(dP_{(x^1,x^2)} - dP^1_{x^1} dP^2_{x^2} \right)$.

Let $\mathbb{X}_n = (X_i)_{1 \le i \le n}$, where $X_i = \left(X_i^1, X_i^2\right)$ i.i.d. $\sim P$ in $\mathcal{X} \times \mathcal{X}$.

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Let $h_\varphi\left((x^1, x^2), (y^1, y^2)\right) = \varphi(x^1, x^2) + \varphi(y^1, y^2) - \varphi(x^1, y^2) - \varphi(y^1, x^2)$,

the previous estimator becomes $\displaystyle \frac{1}{2n(n-1)} \sum_{i \neq j} h\left(X_i, X_j\right).$

Test Statistic

$$\sqrt{n} U_{n,h}\left(\mathbb{X}_n\right) = \frac{\sqrt{n}}{n(n-1)} \sum_{i \neq j} h\left(X_i, X_j\right),$$

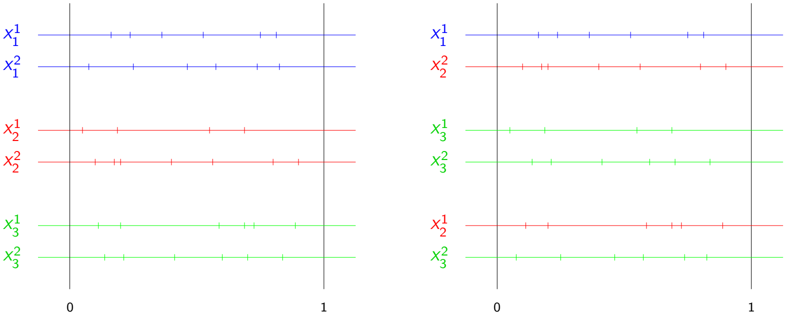
for $h : (\mathcal{X} \times \mathcal{X})^2 \to \mathbb{R}$ well chosen.

Bootstrap approach of Romano (1988)

Given $\mathbb{X}_n = (X_i)_{1\leq i\leq n}$, where $X_i = \left(X_i^1, X_i^2\right)$ i.i.d. $\sim P$ in $\mathcal{X}\times\mathcal{X}$.

The bootstrap sample is

$$\mathbb{X}_n^* = \left(X_{n,1}^*, \ldots, X_{n,n}^*\right) \text{ i.i.d. } \sim P_n^1 \otimes P_n^2 = \left(\frac{1}{n}\sum_{1\leq i\leq n}\delta_{X_i^1}\right)\otimes\left(\frac{1}{n}\sum_{1\leq j\leq n}\delta_{X_j^2}\right).$$



$\sqrt{n}U_{n,h}\left(\mathbb{X}_n\right)$: on original data $\quad$ $\sqrt{n}U_{n,h}\left(\mathbb{X}_n^*\right)$: on bootstrapped data

Theorem 1

Let $h : (\mathcal{X} \times \mathcal{X})^2 \to \mathbb{R}$ such that $(\mathcal{A}_{Exp}^{Cent})$, $(\mathcal{A}_{Emp}^{Cent})$, $(\mathcal{A}_{brk}^{Mmt})$ and $(\mathcal{A}^{Cont})$ hold, then

$$d_{W_2}\left(\mathcal{L}\left(\sqrt{n}U_{n,h}; P_n^1 \otimes P_n^2 | \mathbb{X}_n\right), \mathcal{L}\left(\sqrt{n}U_{n,h}; P^1 \otimes P^2\right)\right) \underset{n\to\infty}{\longrightarrow} 0$$

P-a.s. in $(X_n)_n$.

Here, $\mathcal{L}\left(\sqrt{n}U_{n,h}; Q\right)$ denotes the distribution of $\sqrt{n}U_{n,h}\left(\mathbb{Z}_n\right)$ for $\mathbb{Z}_n = (Z_1, \ldots, Z_n)$ sample of i.i.d. random variables with distribution Q.

Bootstrap test of independence

$$\Phi^{+}_{h,\alpha} = \mathbb{1}_{\left\{\sqrt{n}U_{n,h}(\mathbb{X}_n) > q^*_{h,1-\alpha}(\mathbb{X}_n)\right\}}$$

where $q^*_{h,1-\alpha}(\mathbb{X}_n)$ is the (1-α)-quantile of $\mathcal{L}\left(\sqrt{n}U_{n,h}; P^1_n \otimes P^2_n | \mathbb{X}_n\right)$.

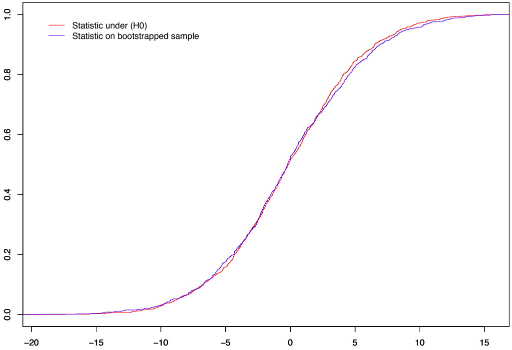
Theorem 2

- *Asymptotic level:*
  Under (H_0),
  $$\mathbb{P}\left(\Phi^{+}_{h,\alpha} = 1\right) \underset{n\to\infty}{\longrightarrow} \alpha$$
- *Asymptotic power:*
  For any alternative P such that $\mathbb{E}\left[\ h(X_i, X_{i'})\ \right] > 0$,
  $$\mathbb{P}\left(\Phi^{+}_{h,\alpha} = 1\right) \underset{n\to+\infty}{\longrightarrow} 1$$

Corollary of Theorem 1

Under (H_0), $P = P^1 \otimes P^2$, and P-a.s. in $(X_n)_n$,

$$d_{W_2}\left(\mathcal{L}\left(\sqrt{n}U_{n,h}; P_n^1 \otimes P_n^2 | \mathbb{X}_n\right), \mathcal{L}\left(\sqrt{n}U_{n,h}; P\right)\right) \underset{n\to\infty}{\longrightarrow} 0.$$



Corollary of Theorem 1

Under (H_0), $P = P^1 \otimes P^2$, and P-a.s. in $(X_n)_n$,

$$d_{W_2}\left(\mathcal{L}\left(\sqrt{n}U_{n,h};P_n^1 \otimes P_n^2 | \mathbb{X}_n\right), \mathcal{L}\left(\sqrt{n}U_{n,h};P\right)\right) \underset{n\to\infty}{\longrightarrow} 0.$$

Centered Gaussian limit of $\sqrt{n}U_{n,h}$ under independence

Under (H_0),

$$\mathcal{L}\left(\sqrt{n}U_{n,h};P\right) \underset{n\to\infty}{\Longrightarrow} \mathcal{N}\left(0,\sigma_P^2\right).$$

Quantiles

$q^*_{h,1-\alpha}(\mathbb{X}_n)$ converges a.s to the (1-α)-quantile of $\mathcal{N}\left(0,\sigma_P^2\right)$.

Finally,

$$\mathbb{P}\left(\Phi^+_{h,\alpha}=1\right) = \mathbb{P}\left(\sqrt{n}U_{n,h}(\mathbb{X}_n) > q^*_{h,1-\alpha}(\mathbb{X}_n)\right) \underset{n\to\infty}{\longrightarrow} \alpha.$$

- LLN for U-statistics :
  if $\mathbb{E}\left[\,|h(X,X')|\,\right] < +\infty$, then

$$U_{n,h}(\mathbb{X}_n) \underset{n\to\infty}{\longrightarrow} \mathbb{E}\left[\ h(X,X')\ \right] > 0, \quad P\text{-a.s. in } (X_n)_n.$$

- Using the convergence of the quantile,

$$\frac{q^*_{h,1-\alpha}(\mathbb{X}_n)}{\sqrt{n}} \underset{n\to\infty}{\longrightarrow} 0.$$

So finally,

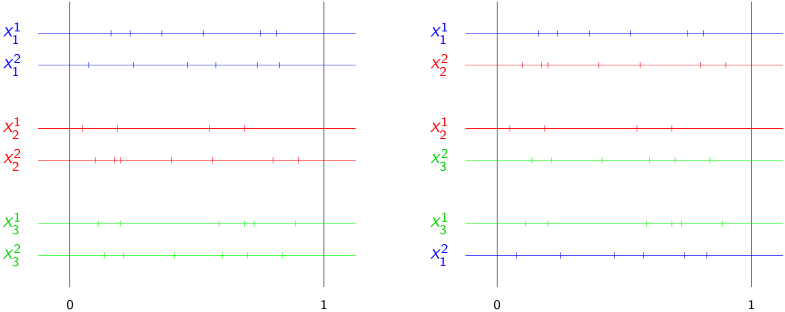
$$\mathbb{P}\left(\Phi^+_{h,\alpha} = 1\right) = \mathbb{P}\left(U_{n,h}(\mathbb{X}_n) > \frac{q^*_{h,1-\alpha}(\mathbb{X}_n)}{\sqrt{n}}\right) \underset{n\to+\infty}{\longrightarrow} 1.$$

Permutated sample

Given $\mathbb{X}_n = (X_i)_{1 \leq i \leq n}$, where $X_i = \left(X_i^1, X_i^2\right)$ i.i.d. $\sim P$ in $\mathcal{X} \times \mathcal{X}$.

The permuted sample is

$$\mathbb{X}_n^{\mathfrak{s}_n} = (X_1^{\mathfrak{s}_n}, \dots, X_n^{\mathfrak{s}_n}), \quad \text{with } X_i^{\mathfrak{s}_n} = \left(X_i^1, X_{\mathfrak{s}_n(i)}^2\right).$$



$\sqrt{n}U_{n,h}(\mathbb{X}_n)$: on original data

$\sqrt{n}U_{n,h}(\mathbb{X}_n^{\mathfrak{s}_n})$: on permuted data (with $\mathfrak{s}_n \sim \mathcal{U}(\mathfrak{S}_n)$ independent of $\mathbb{X}_n$)

Permutated sample

Given $\mathbb{X}_n = (X_i)_{1 \leq i \leq n}$, where $X_i = \left(X_i^1, X_i^2\right)$ i.i.d. $\sim P$ in $\mathcal{X} \times \mathcal{X}$.

$$\mathbb{X}_n^{\mathfrak{s}_n} = \left(X_1^{\mathfrak{s}_n}, \ldots, X_n^{\mathfrak{s}_n}\right), \quad \text{with } X_i^{\mathfrak{s}_n} = \left(X_i^1, X_{\mathfrak{s}_n(i)}^2\right).$$

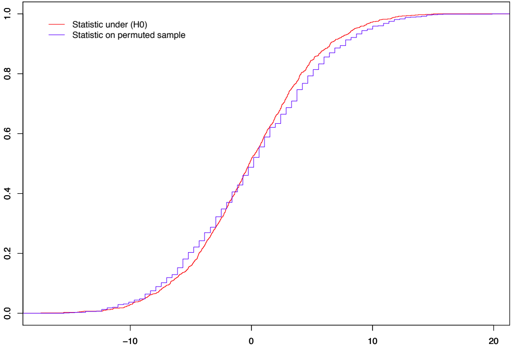
Consistency

Let $\mathfrak{s}_n \sim \mathcal{U}(\{1, \ldots, n\}) \perp\!\!\!\perp \mathbb{X}_n$.
Under (H_0), $\mathbb{X}_n^{\mathfrak{s}_n}$, and $\mathbb{X}_n$ have the same distribution.

Theorem 2

Let $h = h_\varphi$ with $\varphi : \mathcal{X}^2 \to \mathbb{R}$ satisfying $(\mathcal{A}^{Maj,\varphi})$ and $(\mathcal{A}^{Mmt}_{Card})$. Then, under (H_0),

$$d_{W_2}\left(\mathcal{L}\left(\sqrt{n}U_{n,h_\varphi}\left(\mathbb{X}_n^{\mathfrak{s}}\right)|\mathbb{X}_n\right), \mathcal{L}\left(\sqrt{n}U_{n,h}; P^1 \otimes P^2\right)\right) \xrightarrow[n\to\infty]{\mathbb{P}} 0.$$



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$$\Psi^+_{h,\alpha} = \mathbb{1}_{\left\{\sqrt{n}U_{n,h_\varphi}(\mathbb{X}_n) > q^\pi_{h_\varphi,1-\alpha}(\mathbb{X}_n)\right\}}$$

where $q^\pi_{h_\varphi,1-\alpha}(\mathbb{X}_n)$ is the (1-α)-quantile of $\mathcal{L}\left(\sqrt{n}U_{n,h_\varphi}\left(\mathbb{X}_n^{\mathfrak{s}}\right)|\mathbb{X}_n\right)$.

Theorem 2

- *Asymptotic level:*
  Under (H_0),

$$\mathbb{P}\left(\Psi^+_{h_\varphi,\alpha} = 1\right) \le \alpha,$$

  and

$$\mathbb{P}\left(\Psi^+_{h_\varphi,\alpha} = 1\right) \underset{n\to\infty}{\longrightarrow} \alpha.$$

- *Asymptotique power:*
  For any alternative P such that $\mathbb{E}\left[\ h\left(X_i, X_{i'}\right)\ \right] > 0$,

$$\mathbb{P}\left(\Psi^+_{h_\varphi,\alpha} = 1\right) \underset{n\to+\infty}{\longrightarrow} 1.$$

Merci de votre attention.