Convergence of Multivariate Quantile Surfaces

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Outline

Mutidimension

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- Calculability and Fast simulation

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- Here we focus on the halfspace class

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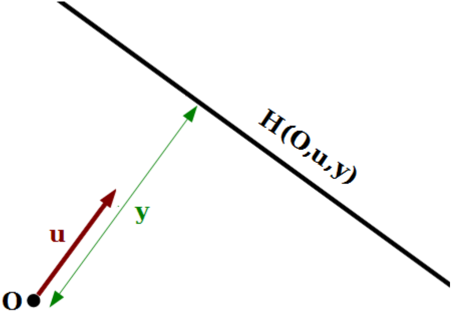
Definitions

Let define the α-th quantile surfaces associated to P and seen from $O \in \mathbb{R}^d$.

$\mathcal{H}$ the collection of all half-spaces and $\langle \cdot, \cdot \rangle$ the inner product in $\mathbb{R}^d$ and

$$H(O,u,y) = \left\{ x \in \mathbb{R}^d : \langle x - O, u \rangle \leq y \right\} \in \mathcal{H}$$

the halfspace standing at distance $y \in \mathbb{R}$ from O in direction $u \in \mathbb{S}_{d-1}$.



Given $\alpha \in (1/2, 1)$ and a direction $u \in \mathbb{S}_{d-1}$ let

$$Y(O, u, \alpha) = \inf \{y : P(H(O, u, y)) \geq \alpha\}$$

be the u-directional α-th quantile range

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$$H_\alpha(u) = H(O, u, Y(O, u, \alpha))$$

be the u-directional α-th quantile halfspace.

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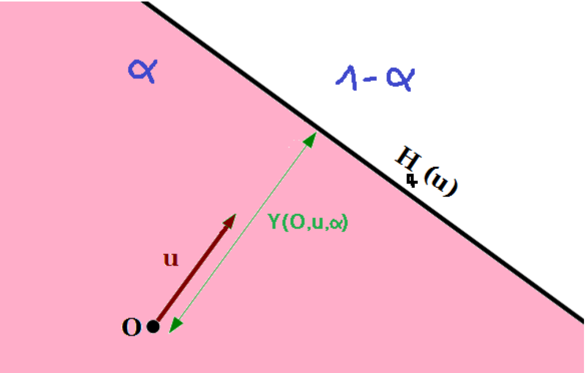


Remark that $Y(O, u, \alpha) = F^{-1}_{\langle X - O, u\rangle}(\alpha)$ and thus $Y(O, u, \alpha)$ is the α-th real quantile of the real random variable $\langle X - O, u\rangle$

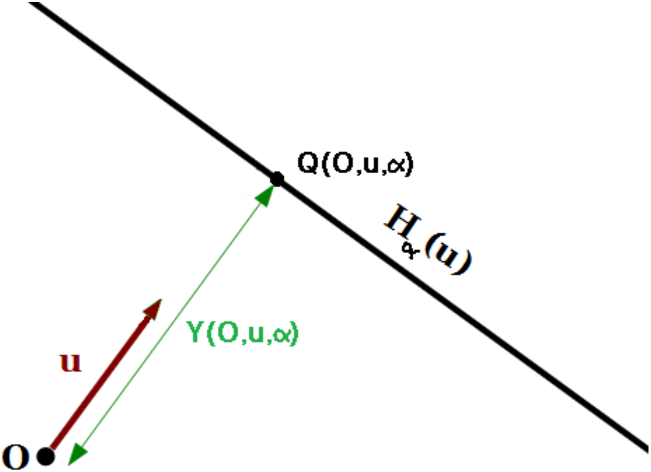
Definition (Multivariate quantile set)

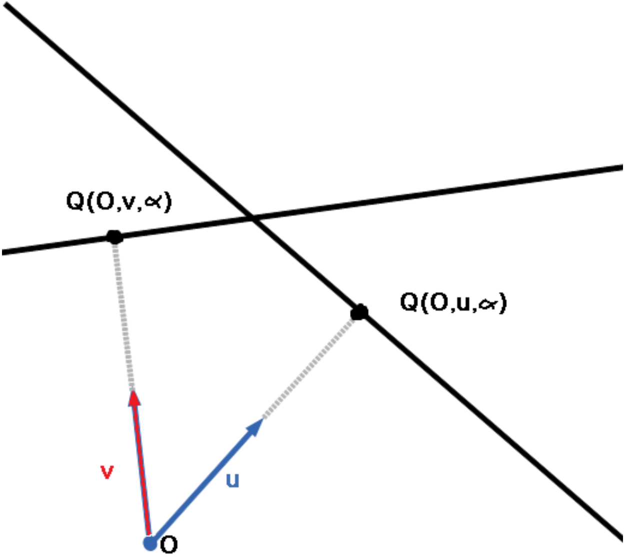
For $\alpha \in (1/2, 1]$, $O \in \mathbb{R}^d$ and $u \in S_{d-1}$ define the u-directional α-th quantile point seen from O to be

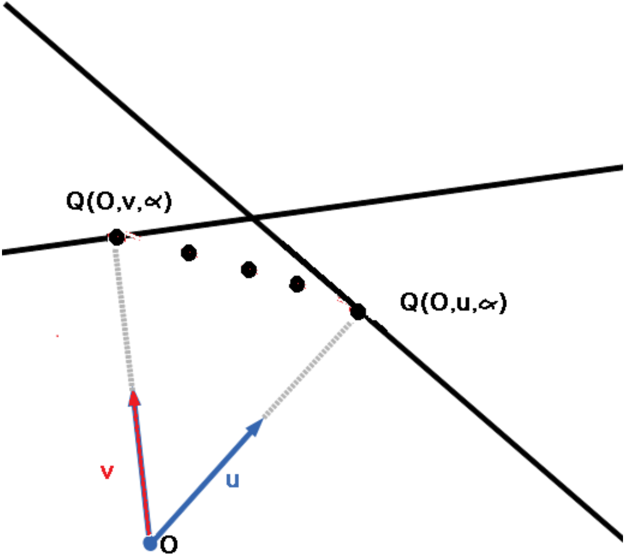
$$Q(O, u, \alpha) = O + Y(O, u, \alpha)u \tag{1}$$

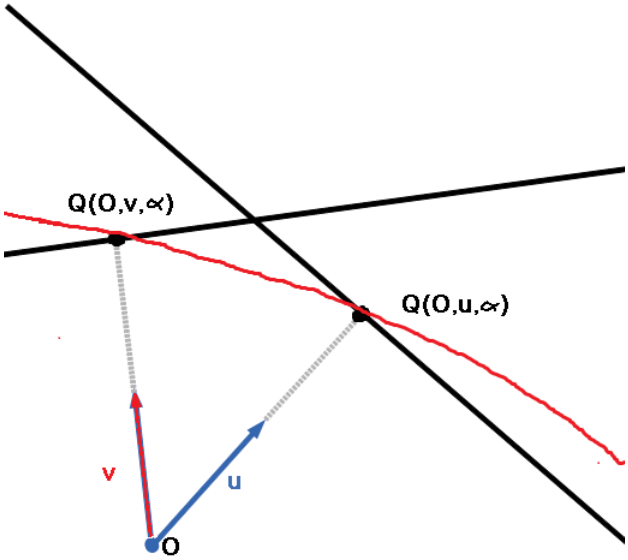
and the α-th quantile set seen from O to be the star-shaped collection of points

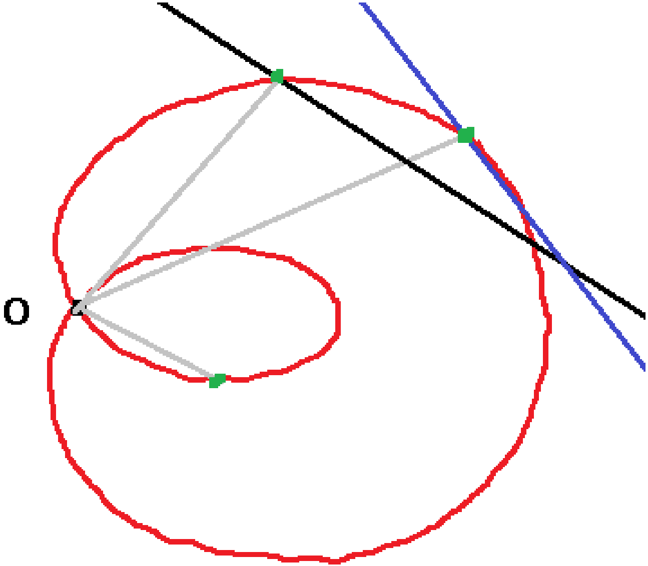
$$\mathbf{Q}_\alpha(O) = \{Q(O, u, \alpha) : u \in \mathbb{S}_{d-1}\}.$$


Q(O,u,α)
H_α(u)
Y(O,u,α)
u
O


Q(O,v,α)
Q(O,u,α)
v
u
O


Q(O,v,α)
Q(O,u,α)
v
u
O


Q(O,v,α)
Q(O,u,α)
v
u
O


O

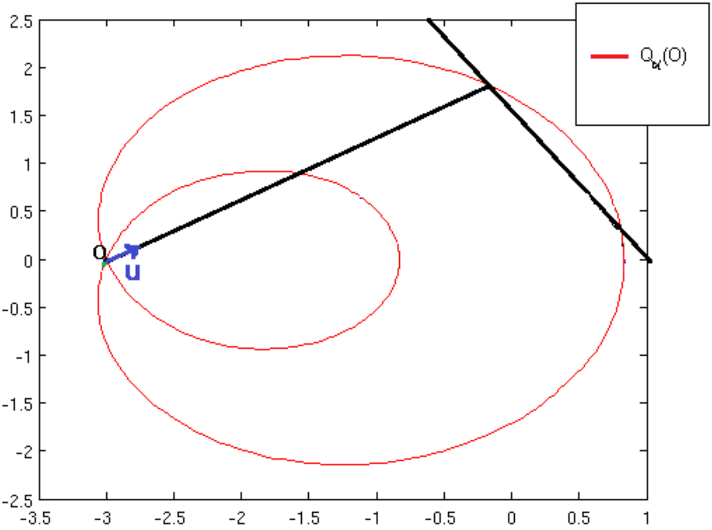
Definition

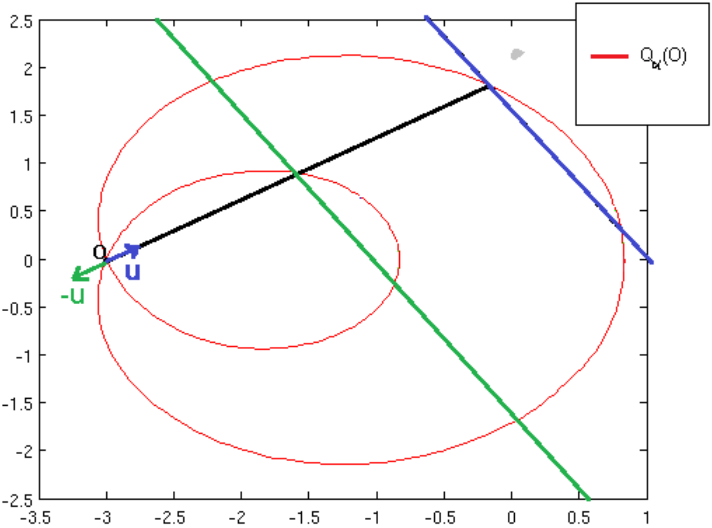
Let

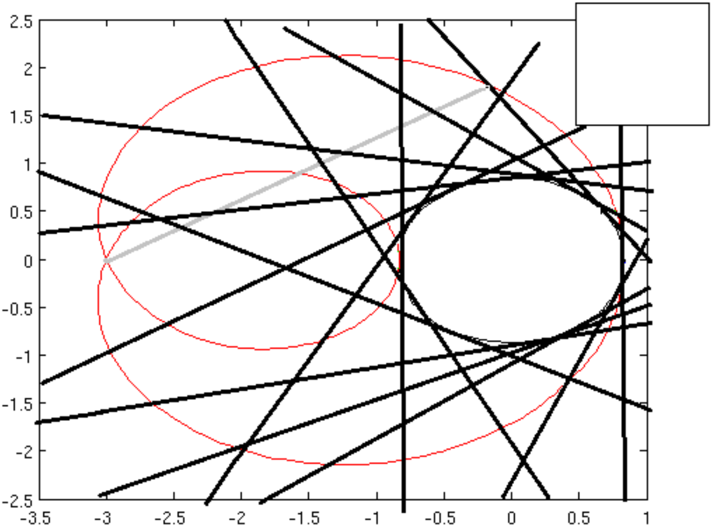
$$\begin{aligned}\mathcal{H}_\alpha &= \{H_\alpha(u) : u \in \mathbb{S}_{d-1}\} \\ &= \{H : H \text{ is a half-space, } P(H) = \alpha\}\end{aligned}$$

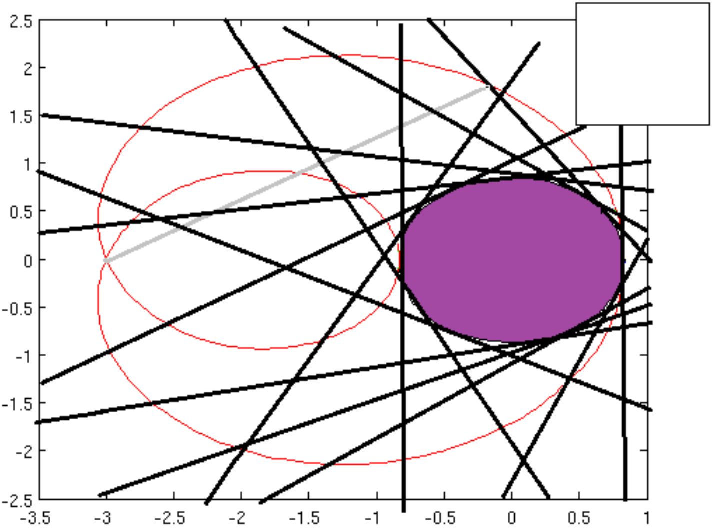
Define

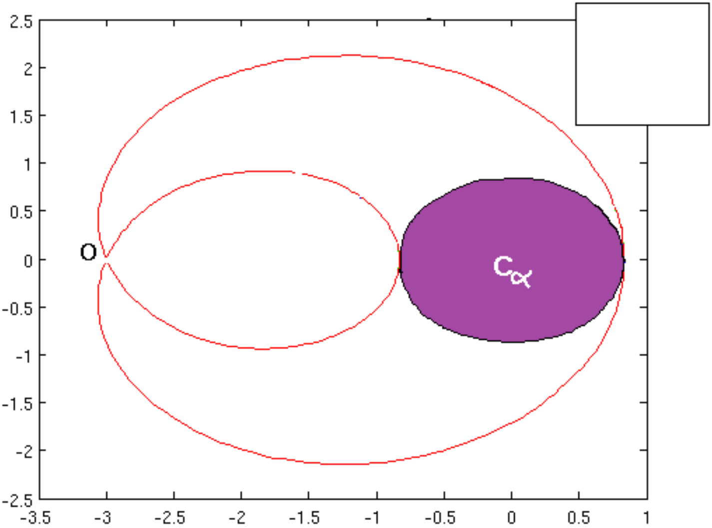
$$\mathcal{C}_\alpha = \bigcap_{H \in \mathcal{H}_\alpha} H.$$

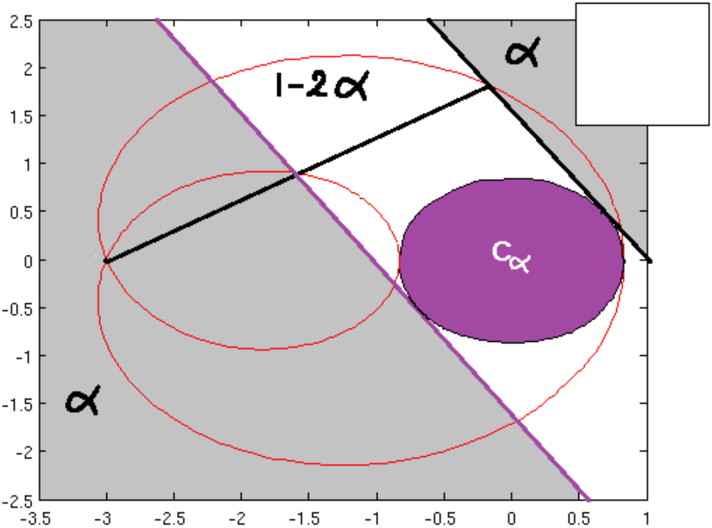

$Q_\alpha(O)$
O
u
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-3.5
-3
-2.5
-2
-1.5
-1
-0.5
0
0.5
1


O
u
-u
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-3.5
-3
-2.5
-2
-1.5
-1
-0.5
0
0.5
1

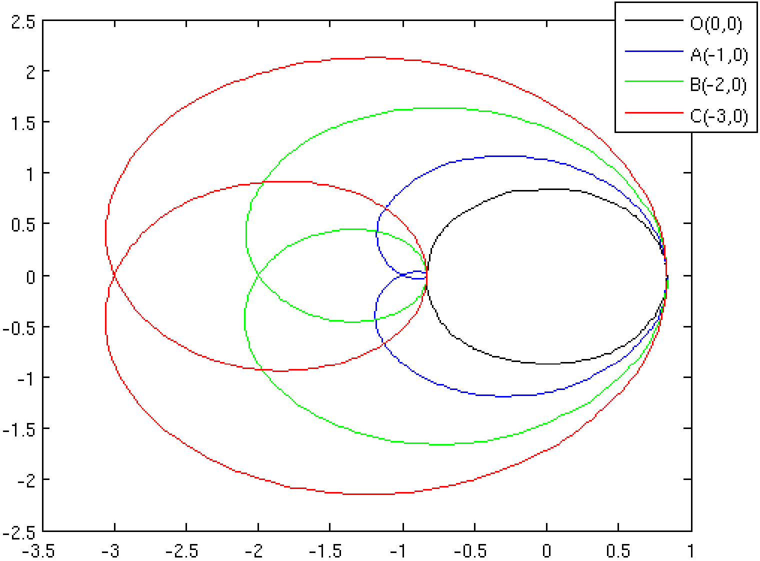

2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-3.5
-3
-2.5
-2
-1.5
-1
-0.5
0
0.5
1

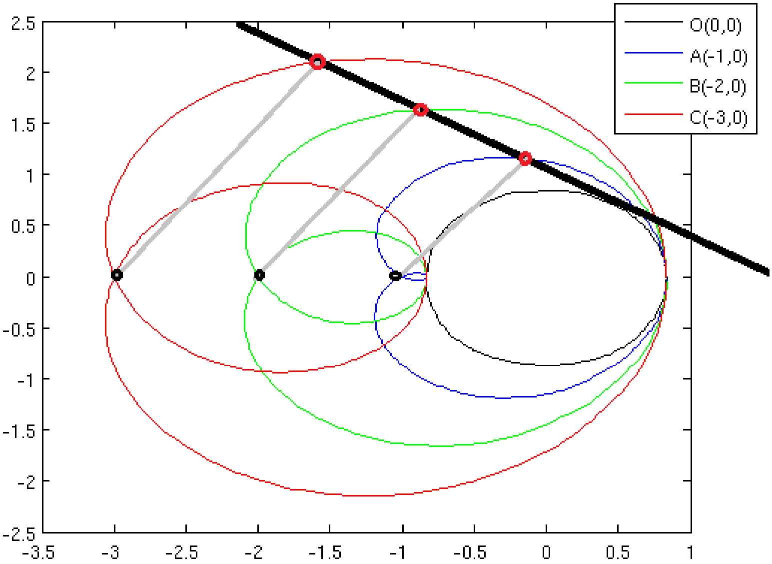

2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-3.5
-3
-2.5
-2
-1.5
-1
-0.5
0
0.5
1


2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-3.5
-3
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
O
C_α



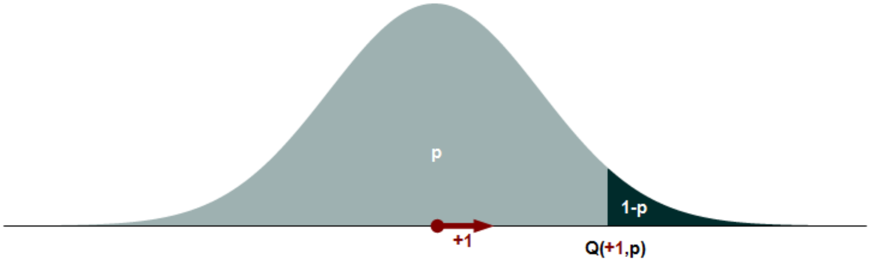
α
1-2α
Cα
α
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-3.5
-3
-2.5
-2
-1.5
-1
-0.5
0
0.5
1

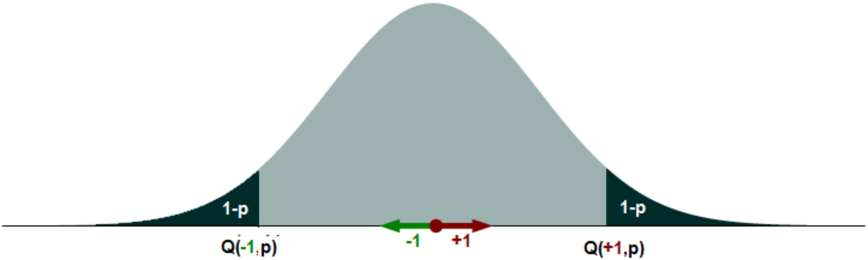

O(0,0)
A(-1,0)
B(-2,0)
C(-3,0)
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-3.5
-3
-2.5
-2
-1.5
-1
-0.5
0
0.5
1

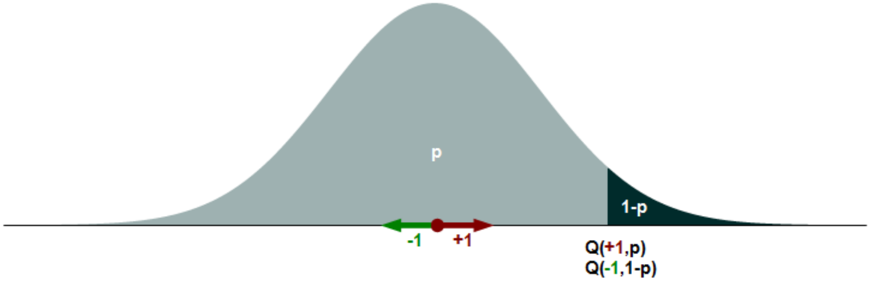

O(0,0)
A(-1,0)
B(-2,0)
C(-3,0)
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-3.5
-3
-2.5
-2
-1.5
-1
-0.5
0
0.5
1

Back to dim $= 1$

In $\mathbb{R}$ we have only two possible directions, $u = \pm 1$


p
1-p
+1
Q(+1,p)


1-p
1-p
Q(-1,p)
-1
+1
Q(+1,p)


p
1-p
-1
+1
Q(+1,p)
Q(-1,1-p)

Outline

Empirical quantile surfaces

Let $\alpha \in \Delta = [\alpha^-, \alpha^+] \subset (1/2, 1)$ and $O \in \mathbb{R}^d$. Define P_n and $P_{n,O,u}$ as follows,

$$P_n = \frac{1}{n} \sum_{i \le n} \delta_{X_i}, \quad P_{n,O,u} = \frac{1}{n} \sum_{i \le n} \delta_{\langle X_i - O, u \rangle},$$

where δ_x is the Dirac mass at $x \in \mathbb{R}^d$ or $x \in \mathbb{R}$.

Definition

For $u \in \mathbb{S}_{d-1}$ let

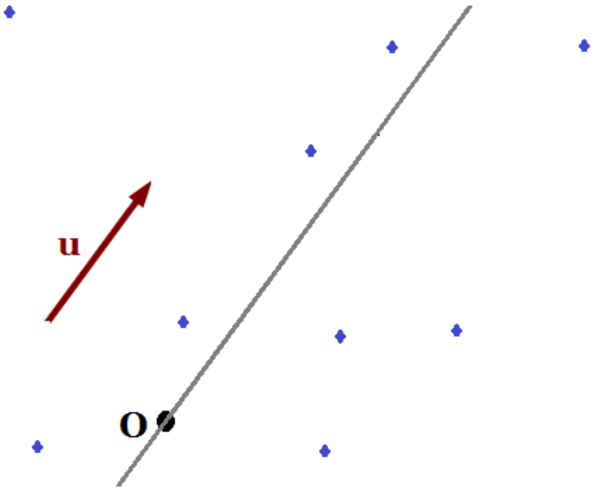
$$\begin{aligned} Y_n(O,u,\alpha) &= \inf\{y : P_n(H(O,u,y)) \geq \alpha\} \\ &= \inf\{y : P_{n,O,u}((-\infty,y)) \geq \alpha\}. \end{aligned}$$

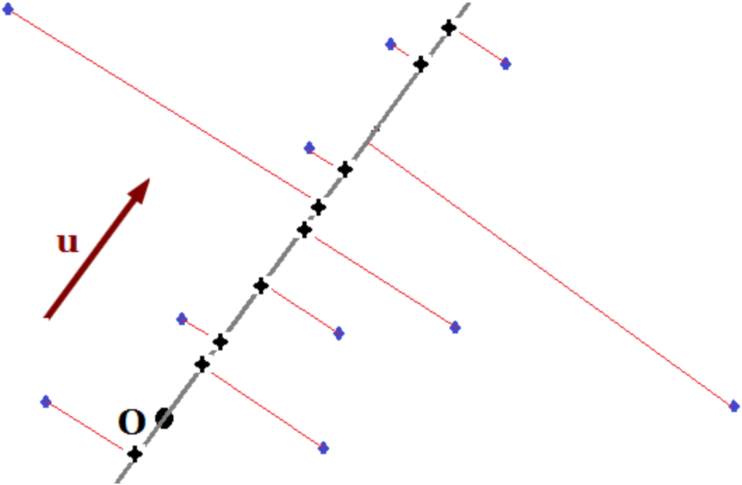
Define the u-directional α-th empirical quantile point seen from O to be

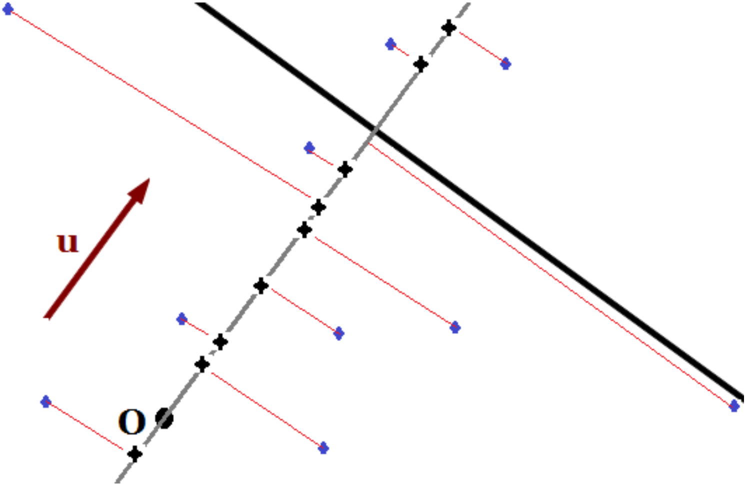
$$Q_n(O,u,\alpha) = O + Y_n(O,u,\alpha)u$$

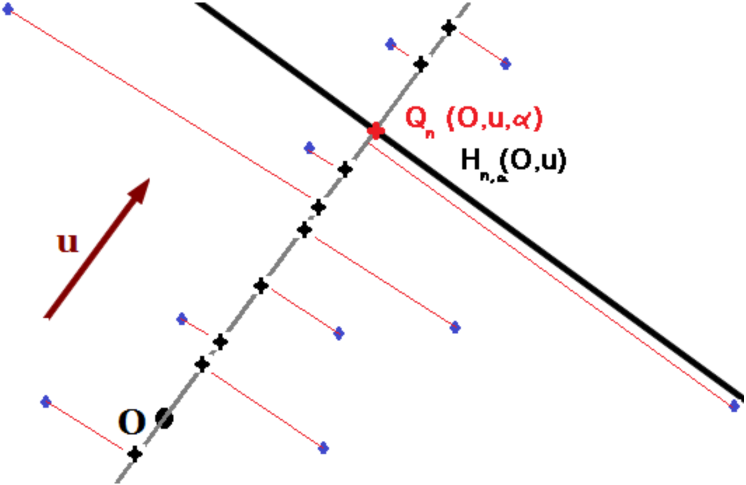
and associate to this point the subjective α-th quantile half-space

$$H_{n,\alpha}(u) = H(O,u,Y_n(O,u,\alpha)).$$


u
O


u
O


u
O



Q_n (O,u,α)
H_n,α(O,u)
u
O

...

Let the α-th empirical quantile set seen from O be

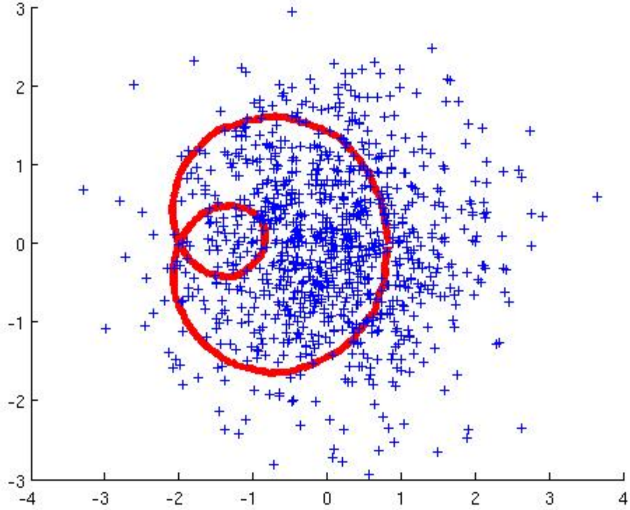
$$Q_{n,\alpha}(O) = \{Q_n(O, u, \alpha) : u \in \mathbb{S}_{d-1}\}.$$

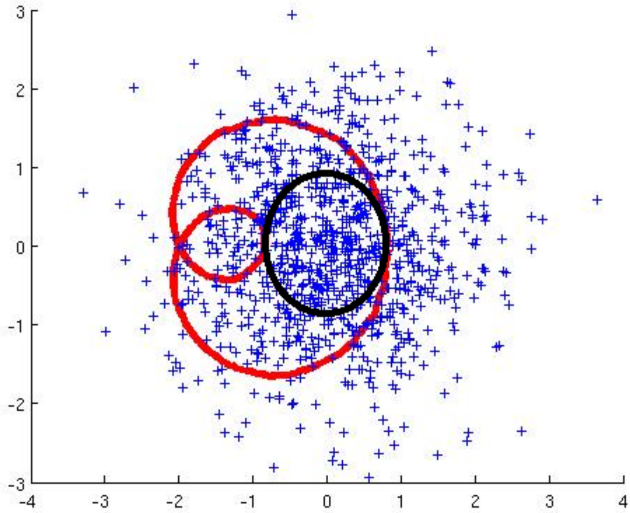
The half-spaces indexed by points $Q_n(O, u, \alpha)$ are collected into

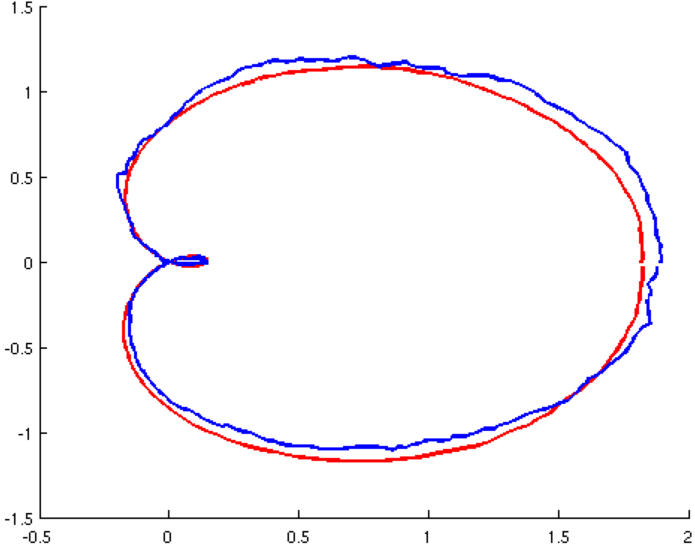
$$\mathcal{H}_{n,\alpha} = \{H_{n,\alpha}(u) : u \in \mathbb{S}_{d-1}\}$$

from which we further define the random convex set

$$\mathcal{C}_{n,\alpha} = \bigcap_{H \in \mathcal{H}_{n,\alpha}} H.$$


3
2
1
0
-1
-2
-3
-4
-3
-2
-1
0
1
2
3
4

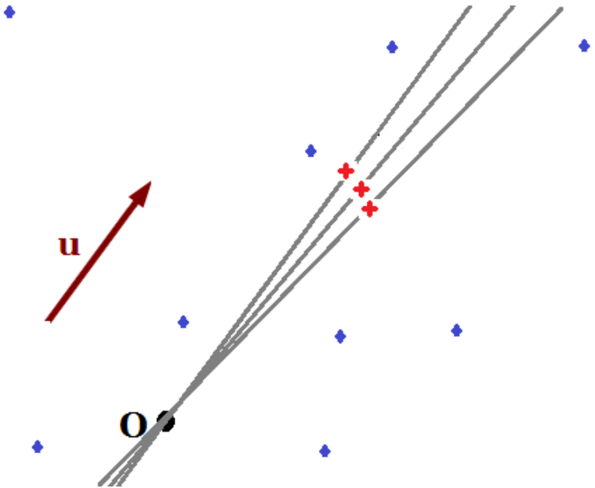

3
2
1
0
-1
-2
-3
-4
-3
-2
-1
0
1
2
3
4


1.5
1
0.5
0
-0.5
-1
-1.5
-0.5
0
0.5
1
1.5
2

Proposition

The sets $\partial\mathcal{C}_{n,\alpha}$ and $Q_{n,\alpha}(O)$ are closed surfaces and piecewise spherical. We also have

$$\mathcal{H}_{n,\alpha} = \left\{ H : H \text{ is a half-space, } P_n(H) \in \left[\alpha, \alpha + \frac{d}{n}\right] \right\}.$$


u
O

Proposition

The processes $Q_n - Q$ and $Y_n - Y$ **do not depend on** *O and*

$$\sqrt{n}(Q_n - Q)(u, \alpha) = \sqrt{n}(Y_n - Y)(u, \alpha) \cdot u \qquad (2)$$

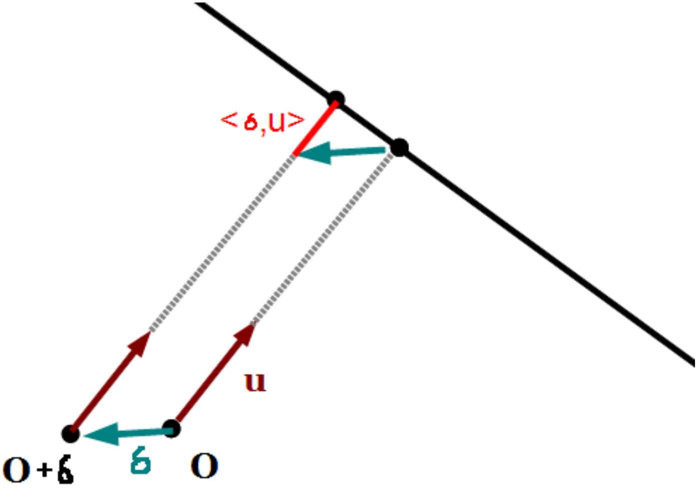
is a $\mathbb{R}^d$-valued empirical process indexed by $S_{d-1} \times \Delta$.

when O moves we have

$$Q(O + \delta, u, \alpha) = Q(O, u, \alpha) + \delta - \langle \delta, u \rangle \, u$$

and

$$Q_n(O + \delta, u, \alpha) = Q_n(O, u, \alpha) + \delta - \langle \delta, u \rangle \, u$$


<δ,u>
u
O+δ
δ
O

Outline

Outline

Directional regularity assumptions

Let $\Delta = [\alpha^-, \alpha^+] \subset \Delta_0 = \left(\alpha_0^-, \alpha_0^+\right)$, where $1/2 < \alpha_0^- < \alpha_0^+ < 1$.

Assumption A0

(A_0): for all $u \in \mathbb{S}_{d-1}$, $F^{-1}_{\langle X-O,u\rangle}$ is continuous on Δ_0

(A_0) implies that the the bands lying between two paralel half-spaces with probability in Δ_0 has itself a strictly positif probability.

Proposition

Assume that $d > 1$ *and* (A_0) *holds. Then the sets* $Q_\alpha(O)$, $\alpha \in \Delta$, *are closed surfaces with at most two connex components. Moreover* $(u, \alpha) \mapsto Q(u, \alpha)$ *is continuous on* $\mathbb{S}_{d-1} \times \Delta$.

From now we suppose for all $u \in \mathbb{S}_{d-1}$, the random variable $\langle X, u\rangle$ has a density $f_{\langle X,u\rangle}$ on $F^{-1}_{\langle X,u\rangle}(\Delta_0)$.

Notation

$$h_u = h_{\langle X,u\rangle} = f_{\langle X,u\rangle} \circ F^{-1}_{\langle X,u\rangle}. \quad (3)$$

Assumption A1

The function $(u,\alpha) \mapsto h(u,\alpha) = h_u(\alpha)$ is continuous on $\mathbb{S}_{d-1} \times \Delta$ and

$$0 < m \leq \inf_{\alpha\in\Delta_0} \inf_{u\in\mathbb{S}_{d-1}} h_u(\alpha) \leq \sup_{\alpha\in\Delta_0} \sup_{u\in\mathbb{S}_{d-1}} h_u(\alpha) \leq M < \infty.$$

Remark:

- Assumption (A_1) **does not imply** that P has a density on $\mathbb{R}^d$. However it implies that $P(\partial H) = 0$ for all $H \in \mathcal{H}$.

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- Under (A_1) $h_u = f_{\langle X-O,u\rangle} \circ F^{-1}_{\langle X-O,u\rangle}$ **do not depend** on O.

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- Allows *supp* P with **low dimension**.

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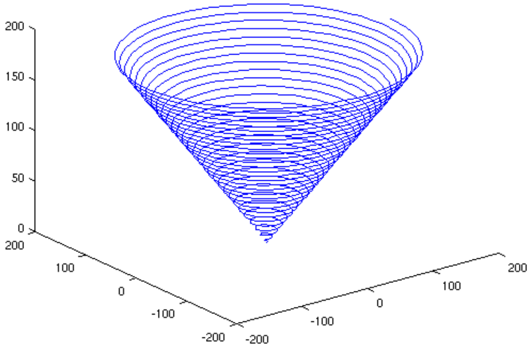

Outline

Main Results

Theorem (Uniform consistency)

Under (A_1) *we have*

$$\lim_{n\to\infty} \sup_{u\in\mathbb{S}_{d-1}} \sup_{\alpha\in\Delta} |Y_n(O,u,\alpha) - Y_\alpha(O,u)| = \lim_{n\to\infty} \|\mathbb{Y}_n - \mathbb{Y}\|_{S_{d-1}\times\Delta} = 0 \;\; a.s.$$

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Corollary

Under (A_1) *we have*

$$\lim_{n\to\infty} \sup_{O\in\mathcal{D}} \sup_{u\in\mathbb{S}_{d-1}} \sup_{\alpha\in\Delta} \|\mathbb{Q}_{\alpha,n}(O,u) - \mathbb{Q}_\alpha(O,u)\| = 0 \quad a.s.$$

$$\lim_{n\to\infty} \sup_{O\in\mathcal{D}} \sup_{\alpha\in\Delta} d_H(\mathbb{Q}_{\alpha,n}(O), \mathbb{Q}_\alpha(O)) = 0 \quad a.s.$$

$$\lim_{n\to\infty} \sup_{\alpha\in\Delta} d_H(\mathcal{C}_{\alpha,n}, \mathcal{C}_\alpha) = 0 \quad a.s.$$

To provide the convergence rate and the confidence region (also the joint covariance) we need central limit theorems.

To provide the UCLT we define the limiting gaussian process $\mathbb{G}_P$

Definition

Let B_P be the P-Brownian bridge indexed by half-spaces, that is the zero mean Gaussian process on $\mathcal{H}$ with
$cov(B_P(H), B_P(H')) = P(H \cap H') - P(H)P(H')$, *for* $(H, H') \in \mathcal{H} \times \mathcal{H}$

To provide the UCLT we define the limiting gaussian process $\mathbb{G}_P$

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For O,u fixed we have

$$cov(B_{O,u}(y), B_{O,u}(y')) = \min(F_{\langle X-O,u\rangle}(y), F_{\langle X-O,u\rangle}(y')) - F_{\langle X-O,u\rangle}(y)F_{\langle X-O,u\rangle}(y').$$

with notation $B_{O,u}(y) := B_P(H(O,u,y))$

Definition

For $(u, \alpha) \in \mathbb{S}_{d-1} \times \Delta$ *we define*

$$\mathbb{G}_P\,(u,\alpha) = \frac{B_P(H(O,u,Y(O,u,\alpha)))}{f_{\langle X,u\rangle} \circ F^{-1}_{\langle X,u\rangle}(\alpha)} = \frac{B_P(H_\alpha(u))}{h(u,\alpha)}. \qquad (4)$$

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We have the covariance

$$cov(\mathbb{G}_P(u_1, \alpha_1), \mathbb{G}_P(u_2, \alpha_2)) = \frac{P(H_{\alpha_1}(u_1) \cap H_{\alpha_2}(u_2)) - \alpha_1\alpha_2}{h(u_1, \alpha_1)h(u_2, \alpha_2)}.$$

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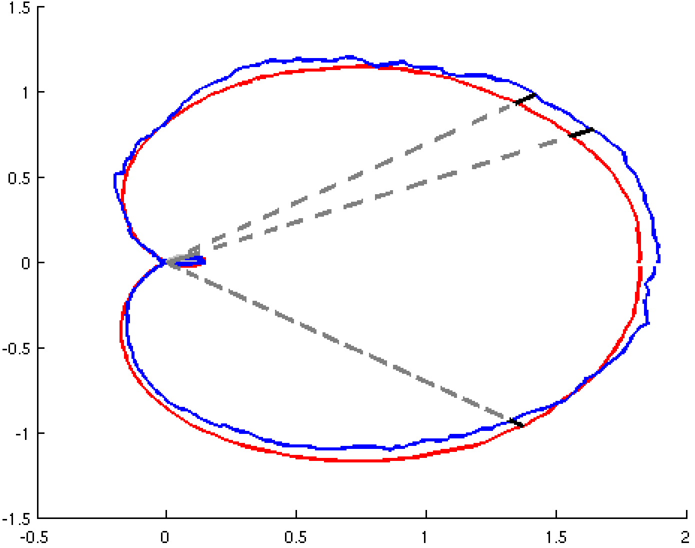
and the correlation

$$corr(\mathbb{G}_P(u_1, \alpha_1), \mathbb{G}_{P,\alpha}(u_2, \alpha_2)) = \frac{P(H_{\alpha_1}(u_1) \cap H_{\alpha_2}(u_2)) - \alpha_1\alpha_2}{\sqrt{\alpha_1(1-\alpha_1)}\sqrt{\alpha_2(1-\alpha_2)}}$$

$$\in \left[-\sqrt{\frac{(1-\alpha_1)(1-\alpha_2)}{\alpha_1\alpha_2}}, \sqrt{\frac{(\alpha_1 \vee \alpha_2)(1-\alpha_1 \wedge \alpha_2)}{(\alpha_1 \wedge \alpha_2)(1-\alpha_1 \vee \alpha_2)}}\right]$$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Let $\mathcal{B}_\infty$ be the set of all bounded real functions on $\mathbb{S}_{d-1} \times \Delta$, endowed with the supremum norm.

Theorem (Uniform central limit theorem)

If P satisfies (A_1) then the sequence $\sqrt{n}(\mathbb{Y}_n - \mathbb{Y})$ weakly converges to $\mathbb{G}_P$ on $\mathcal{B}_\infty$.

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Let remark that:

- we still under (A_1) so P does not need to have a Lebesgue-density!
- the rate of convergence is $\sqrt{n}$ (does not depend on the dimension d !)
- the UCLT here is uniformly on the directions u and the level α

Corollary

Finite dimensional marginal laws convergence Fix $(O_1, u_1, \alpha_1), ..., (O_k, u_k, \alpha_k)$ in $\mathbb{R}^d \times \mathbb{S}_{d-1} \times \Delta$. *Under* (A_1) *we have*

$$\sqrt{n} \begin{pmatrix} Y_n(O_1, u_1, \alpha_1) - Y(O_1, u_1, \alpha_1) \\ ... \\ Y_n(O_k, u_k, \alpha_k) - Y(O_k, u_k, \alpha_k) \end{pmatrix} \to_{law} \mathcal{N}(0_k, \Sigma)$$

where the limiting covariance matrix Σ *has coordinates*

$$\Sigma_{i,j} = \frac{P(H_{\alpha_i}(u_i) \cap H_{\alpha_j}(u_j)) - \alpha_i \alpha_j}{h_{\alpha_i}(u_i) h_{\alpha_j}(u_j)}.$$

Assumption A2

(A_2) :(A_1) holds and $h(u, \alpha) = h_u(\alpha)$ is differentiable on $\mathbb{S}_{d-1} \times \Delta_0$ in variables (u, α) with uniformly bounded derivatives.

Theorem (Uniform strong approximation with rate)

if (A_1) *then one can construct on the same probability space* $(\Omega, \mathcal{T}, \mathbb{P})$ *an i.i.d. sequence* X_n *with law* P *and* $\mathbb{G}_n$ *versions of* $\mathbb{G}_P$ *such that for* $O \in \mathbb{R}^d, \alpha \in \Delta, u \in \mathbb{S}_{d-1}$

$$\mathbb{Y}_n(O, u, \alpha) = \mathbb{Y}(O, u, \alpha) + \frac{\mathbb{G}_n(u, \alpha)}{\sqrt{n}} + \frac{\mathbb{Z}_n(u, \alpha)}{\sqrt{n}} \qquad (5)$$

where $\mathbb{Z}_n = \sqrt{n}(\mathbb{Y}_n - \mathbb{Y}) - \mathbb{G}_n$ *is such that*

$$\lim_{n\to\infty} \|\mathbb{Z}_n\|_{S_{d-1}\times\Delta} = \lim_{n\to\infty} \sup_{u\in\mathbb{S}_{d-1}} \sup_{\alpha\in\Delta} |\mathbb{Z}_n(u, \alpha)| = 0 \quad a.s.$$

...

If P moreover satisfies (A_2) then $\mathbb{G}_n$ can be constructed such that for $v_1 = v_2 = 1/4$, $w_1 = 1/2$, $w_2 > 1$ and, if $d \geq 3$, $v_d = 1/(3+4d)$, $w_d > 1$, there exists $c_\theta(m, M, d) > 0$ and $n_\theta(m, M, d) > 0$ such that we have, for all $n > n_\theta$,

$$\mathbb{P}\left(\|\mathbb{Z}_n\|_{S_{d-1}\times\Delta} \geq c_\theta \frac{(\log n)^{w_d}}{n^{v_d}}\right) \leq \frac{1}{n^\theta}. \tag{6}$$

Let $\Lambda_n = \sqrt{n}(P_n - P)$ be the empirical process indexed by $\mathcal{H}$ and define

$$\mathbb{E}_n(u, \alpha) = \Lambda_n(H_\alpha(u)) = \sqrt{n}(P_n(H_\alpha(u)) - \alpha)$$

its restriction to

$$\mathcal{H}_\Delta = \bigcup_{\alpha \in \Delta} \mathcal{H}_\alpha = \{H : H \text{ is a half-space, } P(H) \in \Delta\}$$

Theorem (Bahadur-Kiefer type representation of multivariate quantiles)

If P satisfies (A_1)then we have

$$\lim_{n\to\infty} \left\| \sqrt{n}(\mathbb{Y}_n - \mathbb{Y})h + \mathbb{E}_n \right\|_{S_{d-1}\times\Delta} = 0 \quad a.s. \tag{7}$$

and if moreover satisfies (A_2) then it holds

$$\left\| \sqrt{n}(\mathbb{Y}_n - \mathbb{Y})h + \mathbb{E}_n \right\|_{S_{d-1}\times\Delta} = O_{a.s.}\left(\frac{(\log n)^{w_d}}{n^{v_d}} \right). \tag{8}$$

Outline

Quantile via General classes

Definition

Let $O \in \mathbb{R}^d$, $u_0 \in \mathbb{S}_{d-1}$ *and* φ *be a continuous function from* $\mathbb{R}^d$ *to* $\mathbb{R}$ *satisfying*

$$\varphi^{-1}((-\infty, y_1]) = G_{y_1} \subset G_{y_2}, \quad y_1 \leq y_2,$$
$$\lambda_d(\varphi^{-1}(\{y\})) = 0, \quad y \in \mathbb{R}.$$

For any $u \in \mathbb{S}_{d-1}$ *write* r_u *the rotation of* $\mathbb{R}^d$ *having center* O_d *and angle* $u_0 \hookrightarrow u$ *and* t_O *the translation directed by* O. *Define*

$$Y(O, u, \alpha) = \inf \{y : P(t_O \circ r_u(G_y)) \geq \alpha\}$$

to be the u-directional (φ, u_0)*-shaped* α*-th quantile range from* O

Definition

and

$$Q_\alpha(O) = \{O + Y(O, u, \alpha)u : u \in \mathbb{S}_{d-1}\}$$

to be the (φ, u_0)*-shaped* α*-th quantile surface seen from* O.

Remark:

The halfspace case is a special case where $\varphi_{u_0}(x) = \langle X, u_0 \rangle$, $G_y = \varphi_{u_0}^{-1}((-\infty, y]) = H(0_d, u_0, y)$.

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The halfspace case is a special case where $\varphi_{u_0}(x) = \langle X, u_0 \rangle$, $G_y = \varphi_{u_0}^{-1}((-\infty, y]) = H(0_d, u_0, y)$.

Let define

$$\mathbb{G}_P(O, u, \alpha) = \frac{B_P(t_O \circ r_u(G_{Y(O,u,\alpha)}))}{h_{O,u}(\alpha)}$$
$$= \frac{B_P(t_O \circ r_u \circ \varphi^{-1}((-\infty, Y(O, u, \alpha)]))}{f_{\varphi \circ r_u^{-1} \circ t_O^{-1}(X)} \circ F^{-1}_{\varphi \circ r_u^{-1} \circ t_O^{-1}(X)}}$$

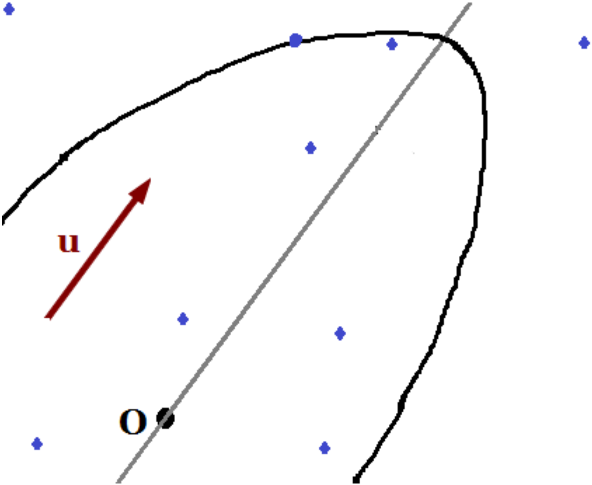
with

$$h_{O,u}(\alpha) = f_{\varphi(r_u^{-1}(X-O))} \circ F^{-1}_{\varphi(r_u^{-1}(X-O))}$$

Let $\mathcal{B}_\infty$ be the set of all bounded real functions on $\mathbb{S}_{d-1} \times \Delta$, endowed with the supremum norm.

Theorem (Uniform central limit theorem)

If P satisfies (B_1) then the sequence $\sqrt{n}(\mathbb{Y}_n - \mathbb{Y})$ weakly converges to $\mathbb{G}_P$ on $\mathcal{B}_\infty$.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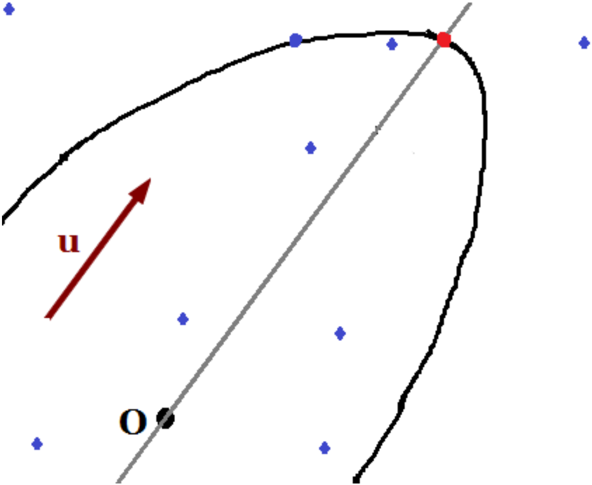
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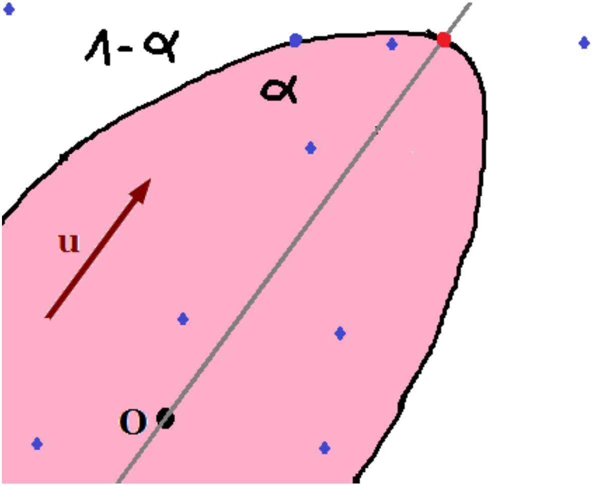
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Outline

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- Joint region of confidence

- Extension to general classes (geodesics,...)

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