Non-parametric Bayesian test for monotonicity

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"We balance probability and choose the most likely. It is the scientific use of the imagination."

-Sherlock Holmes, *The hound of Baskervilles*

Introduction

Consider the usual Gaussian regression setting

$$Y_i = f(i/n) + \sigma\epsilon_i$$

where $\epsilon_i \overset{iid}{\sim} \mathcal{N}(0,1)$. We want to test whether f satisfy some **shape constraints** (such as monotonicity).

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- Drug response models
- Temperatures anomalies problems
- Economic models when assessing a global trend

Introduction cnt'd

Let $\mathcal{F}$ be the set of **uniformly bounded monotone non increasing functions**. We want to test

$$H_0 : f \in \mathcal{F}, \text{ versus } H_1 : f \notin \mathcal{F}$$

This problem has already been address in the frequentist literature

- Baraud et al. (2005) use projection of the regression function on the set of piecewise constant functions,
- Hall and Heckman (2000) and Ghosal et al. (2000) test negativity of the derivative of the regression function,
- Akakpo et al. (2012) propose a procedure that detect departure from monotonicity via local least concave majorant.

Introduction cnt'd

Remarks

- These methods require in general heavy computations to be used in practice,
- There is only one other (to my knowledge) Bayesian procedure to test for monotonicity.

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Introduction cnt'd

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We thus want to construct a Bayesian testing procedure...

- ... that is easy to implement/use in practice
- ... that is consistent
- ... that achieve the optimal separation rate

Prior construction

We consider the following prior

$$\Pi : \begin{cases} k \sim \pi_k \\ \sigma \sim \pi_\sigma \\ \omega_1 \dots \omega_k | k \sim \otimes_{i=1}^{n} \pi_\omega \end{cases}$$

and f is defined by

$$f_{\omega,k}(\cdot) = \sum_{i=1}^{k} \omega_i \mathbb{I}_{[(i-1)/k, i/k)}(\cdot)$$

We thus have a posterior distribution for f

$$\Pi(f, \sigma | \mathbf{Y}) \propto \Pi(f, \sigma) \mathcal{L}(\mathbf{Y}, f, \sigma)$$

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$$\Pi(f, \sigma | \mathbf{Y}) \propto \Pi(f, \sigma) \prod_{i=1}^{n} \sigma^{-1} \phi\left(\frac{Y_i - f(i/n)}{\sigma}\right)$$

Bayes Factor

The standard Bayesian approach will be to compute the Bayes Factor We thus consider the Bayes Factor

$$B_{0,1} = \frac{\Pi(f \in \mathcal{F}|\mathbf{Y}) \diagup \Pi(\mathcal{F})}{\Pi(f \notin \mathcal{F}|\mathbf{Y}) \diagup 1 - \Pi(\mathcal{F})}$$

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Drawbacks

- The computation of $\Pi(\mathcal{F})$ is difficult and require some strong conditions on the prior
- This approach does not give satisfactory results
- We could not prove consistency for this method

Alternative approach

We thus consider an alternative test

$$H_0^a : \delta(f, \mathcal{F}) \leq \tau \quad \text{versus} \quad H_1^a : \delta(f, \mathcal{F}) > \tau$$

where $\delta(f, \mathcal{F})$ is a discrepancy measure between f and $\mathcal{F}$ and τ a threshold.

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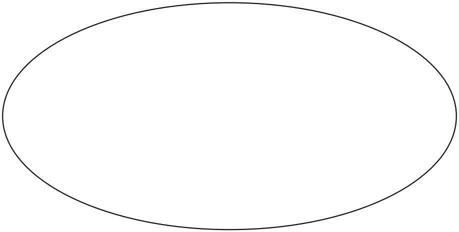

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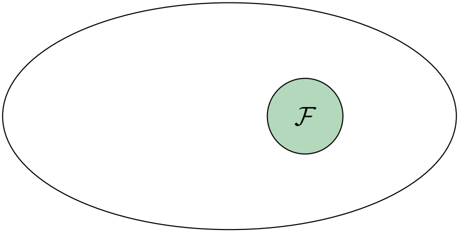

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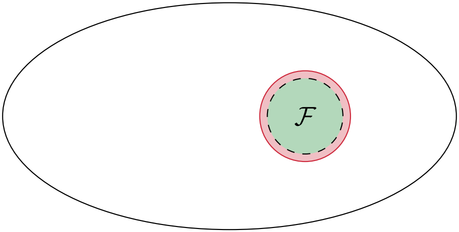

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where $\delta(f, \mathcal{F})$ is a discrepancy measure between f and $\mathcal{F}$ and τ a threshold. We consider the standard $0-1$ loss and thus define our test as

$$\delta_n^\pi(\tau) := \begin{cases} 0 \text{ if } \Pi\left(\delta(f, \mathcal{F}) \leq \tau | \mathbf{Y}\right) \geq \frac{\gamma_0}{\gamma_0+\gamma_1} \\ 1 \text{ otherwise} \end{cases}$$

Alternative approach cnt'd

We want both test to be asymptotically equivalent

- We thus set $\tau = \tau_n \underset{n \to \infty}{\longrightarrow} 0$
- For some distance d and for all $\rho > 0$

$$\sup_{f \in \mathcal{F}} \mathrm{E}_f^n(\delta_n^\pi(\tau_n)) = o(1)$$

$$\sup_{f, d(f, \mathcal{F}) > \rho} \mathrm{E}_f^n(1 - \delta_n^\pi(\tau_n)) = o(1).$$

- We will ask that the minimum value $\rho = \rho_n$ such that the test is consistent is close to the minimax separation rate.

Construction of the test

Recall

$$f_{\omega,k}(\cdot) = \sum_{i=1}^{k} \omega_i \mathbb{I}_{[\frac{i-1}{k},\frac{i}{k}[}(\cdot)$$

We consider for $\delta(f_{\omega,k}, \mathcal{F})$ the supremum norm between $f_{\omega,k}$ and $\mathcal{F}$, more precisely, we take

$$\delta(f_{\omega,k}, \mathcal{F}) = H(\omega, k) = \max_{1 \leq i < j \leq k} (\omega_j - \omega_i)$$

We now need a good calibration of τ_n such that our procedure is consistent and achieve the **optimal separation rate**. We consider Hölderian alternatives with $\alpha \leq 1$

$$f \in \mathcal{H}(\alpha, L) = \left\{f, [0,1] \to \mathbb{R}, \forall x, y \in [0,1]^2 |f(y) - f(x)| \leq L|y - x|^{\alpha}\right\}$$

Conditions

Consider the model $Y_i = f(i/n) + \sigma\epsilon_i$ where $\epsilon_i \sim \mathcal{N}(0,1)$. We take Π of the form

$$d\Pi(f_{\omega,k},\sigma) = \pi_k(k)\pi_\sigma(\sigma)\prod_{i=1}^{k}\pi_\omega(\omega_i)$$

We get these two conditions

- C1 The densities π_ω and π_σ are continuous, $\pi_\omega(x) > 0$ for all $x \in \mathbb{R}$ and $\pi_\sigma(t) > 0$ for all $t \in \mathbb{R}_*^+$,
- C2 π_k is such that there exist positive constants C_d and C_u such that

$$e^{-C_d k L(k)} \leq \pi_k(k) \leq e^{-C_u k L(k)}$$

where $L(k)$ is either $\log(k)$ or 1.

Main Theorem

Theorem

*Under the assumptions **C1** and **C2**, setting* $\tau_n^k = M_0\sqrt{k\log(n)/n}$ *and* δ_n^π *the testing procedure*

$$\delta_n^\pi = \mathbb{I}\left\{\pi\left(H(\omega,k) > \tau_n^k | Y^n\right) > \frac{\gamma_0}{\gamma_1+\gamma_0}\right\}$$

thus

$$\begin{aligned} \sup_{f\in\mathcal{F}} \mathrm{E}_f^n(\delta_n^\pi) = o(1) \\ \sup_{f,d(f,\mathcal{F})>\rho, f\in\mathcal{H}(\alpha,L)} \mathrm{E}_f^n(1-\delta_n^\pi) = o(1) \end{aligned} \quad (1)$$

for all $\rho > \rho_n(\alpha) = M(n/\log(n))^{-\alpha/(2\alpha+1)}v_n$ *where* $v_n = 1$ *when* $L(k) = \log(k)$ *and* $v_n = \sqrt{\log(n)}$ *when* $L(k) = 1$.

Main Theorem

Remarks

- The conditions on the prior are mild and satisfied for a wide variety of distributions
- Neither the prior nor the hyperparameters depend on the regularity under the alternative,
- the separation rate $\rho_n(\alpha)$ is the minimax separation rate (up to a $\log(n)$ factor)

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"Let us hear the suspicions. I will look after the proof."
-Sherlock Holmes, *The Adventure of Three Students*

Simple choice for the prior

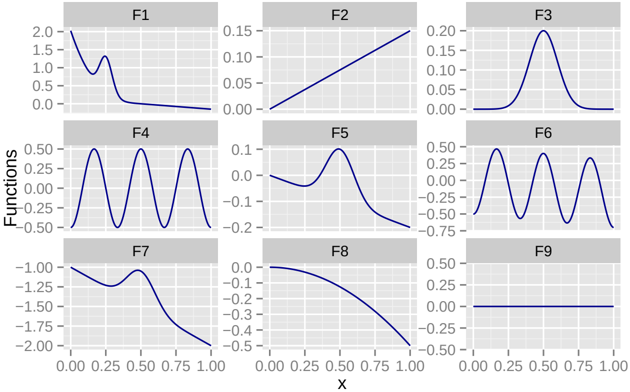
We choose

- $k \sim \mathrm{Geom}(\lambda)$
- $\sigma | k \sim \Gamma^{-1}(a, b)$
- $\omega_i | k, \sigma^2 \sim \mathcal{N}(m, \sigma^2/\mu)$

We also need to calibrate the hyperparameters λ, a, b, m and μ, and the constant M_0 in τ_n^k.

Simulated data

We run our test for nine functions adapted from the frequentist literature



Simulated data

Table : Percentage of rejection for the simulated examples

	f_0	σ^2	Barraud et al. $n = 100$	Akakpo et al. $n = 100$	Bayes Test, n : 100	250	500	1000
H_0	f_1	0.01	99	99	97	100	100	100
	f_2	0.01	99	100	95	100	100	100
	f_3	0.01	99	98	100	100	100	100
	f_4	0.01	100	99	100	100	100	100
	f_5	0.004	99	99	100	100	100	100
	f_6	0.006	98	99	100	100	100	100
	f_7	0.01	76	68	97	100	100	100
H_1	f_8	0.01	-	-	2	0	0	0
	f_9	0.01	-	-	2	3	2	2

Temperature annomalies

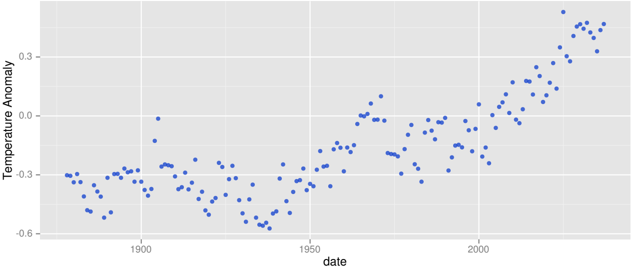


Figure : Temperature Anomalies

We compute $\hat{\Pi}(H(\omega, k) > \tau_n^k | \mathbf{Y}) = 0.98$, and thus **reject** monotonicity of the cuve.

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Conclusion

"It is capital mistake to theorize before one has data."
-Sherlock Holmes, *A Scandal in Bohemia*

We proposed a test for monotonicity in a case where **the standard Bayesain approach fails** Our procedure is

- easy to implement,
- gives good results in practice,
- has good asymptotic properties.

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extentions

- extend to other types of shape constrains
- study how our procedure behaves for non Gaussian errors

Thank You !

Go Bayes !

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