# Quelques exemples de déconvolutions semi-aveugles

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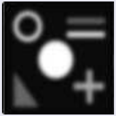
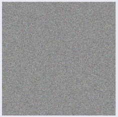
# Framework of inverse problems in statistics

## Goal

Find $\underbrace{\mathbf{f}}$ , given $\underbrace{\mathbf{Y}_\varepsilon} = \underbrace{\mathbf{Kf}} + \underbrace{\varepsilon \mathbf{W}}$





- $\mathbf{f} \in \mathbb{H} = \mathbb{L}^2(\mathcal{Y}, \mu)$.
- $\mathbf{K}$ is a linear operator from $\mathbb{H}$ to $\mathbb{H}$.
- $\mathbf{W}$ is a Gaussian white noise on $\mathbb{H}$

**Example :**

- Deconvolution on $\mathbb{T}^2$ : for $\mathbf{k}, \mathbf{f} \in \mathbb{L}^2(\mathbb{T}^2)$,

$$\mathbf{Kf}(x) = \mathbf{k} * \mathbf{f}(x) = \int_{\mathbb{T}^2} \mathbf{k}(x-t)\mathbf{f}(t)dt \,.$$

# The Gaussian white noise (GWN) model

We observe

$$\mathbf{Y}_\varepsilon = \mathbf{Kf} + \varepsilon \mathbf{W}$$

where $\mathbf{W}$ is a Gaussian white noise (GWN) on $\mathbb{H}$.

## Observable quantities :

For $u \in \mathbb{H}$,

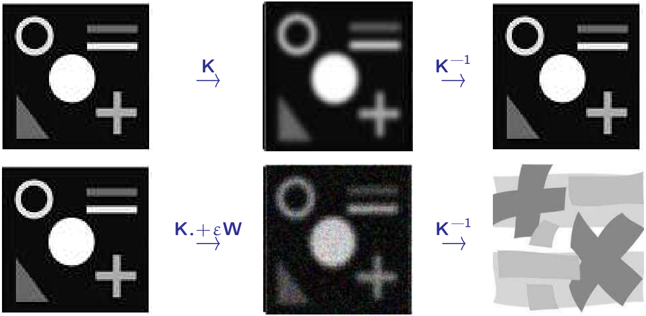
$$\langle \mathbf{Y}_\varepsilon, u \rangle = \langle \mathbf{Kf}, u \rangle + \varepsilon \xi_u \text{ where } \xi_u \sim \mathcal{N}(0, \|u\|^2_{\mathbb{H}})$$

and, for $u, v \in \mathbb{H}$,

$$\mathrm{Cov}\big(\xi_u, \xi_v\big) = \langle u, v \rangle_{\mathbb{H}}.$$

# The problem with inverse problems : Hadamard's conditions of well-posedness

- If $\mathbf{K}$ is bijective and $\mathbf{K}^{-1}$ is continuous, the problem is well-posed.
- If $\mathbf{K}$ is compact, the last assumption fails. The problem is ill-posed.




**Scheme**

Solve $\mathcal{P}_h$ approximating $\mathcal{P}$. The smaller $h$, the more ill-posed $\mathcal{P}_h$.
$\Rightarrow$ There is a balance between accuracy and well-posedness.

# Problem and discretization

## Goal

Recover $\mathbf{f} \in \mathbb{L}^2(\mathbb{T})$ from

$$\mathbf{Y}_\varepsilon = \mathbf{k} * \mathbf{f} + \varepsilon \mathbf{W},$$

where $\mathbf{k} \in \mathbb{L}^2(\mathbb{T})$ and $\mathbf{W}$ is a GWN on $\mathbb{L}^2(\mathbb{T})$.

## Projection on harmonic functions

Let $u_\ell : t \mapsto e^{2i\pi\ell t}$, $\ell \in \mathbb{Z}$.
Then

$$\langle \mathbf{K}\mathbf{f}, u_\ell \rangle = \langle \mathbf{k}, u_\ell \rangle \langle \mathbf{f}, u_\ell \rangle = \mathbf{k}_\ell \mathbf{f}_\ell$$

$$\Rightarrow \mathbf{Y}_{\varepsilon,\ell} = \mathbf{k}_\ell \mathbf{f}_\ell + \varepsilon \mathbf{W}_\ell$$

where $\mathbf{W}_\ell$ are i.i.d $\mathcal{N}(0,1)$.

# Matricial representation

**Signal** : $\mathbf{Y}_\varepsilon = \mathbf{k} * \mathbf{f} + \varepsilon \mathbf{W}$

$$\begin{pmatrix} \mathbf{Y}_{\varepsilon,1} \\ \mathbf{Y}_{\varepsilon,2} \\ \vdots \\ \mathbf{Y}_{\varepsilon,\ell} \end{pmatrix} = \begin{pmatrix} \mathbf{k}_1 & & & \\ & \mathbf{k}_2 & & \\ & & \ddots & \\ & & & \mathbf{k}_\ell \end{pmatrix} \begin{pmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \vdots \\ \mathbf{f}_\ell \end{pmatrix} + \varepsilon \begin{pmatrix} \mathbf{W}_1 \\ \mathbf{W}_2 \\ \vdots \\ \mathbf{W}_\ell \end{pmatrix}, \quad \begin{pmatrix} \mathbf{W}_1 \\ \mathbf{W}_2 \\ \vdots \\ \mathbf{W}_\ell \end{pmatrix} \sim \mathcal{N}(0, \mathbf{I}_\ell)$$

# Algorithm

**Reconstruction formula** :

$$\mathbf{f}_\ell = \mathbf{k}_\ell^{-1} \left[\mathbf{k} * \mathbf{f}\right]_\ell$$

Define the thresholding level :

$$\mathrm{Sig}_\ell = \tau\varepsilon\sqrt{|\log \varepsilon|}\,.$$

Then

$$\tilde{\mathbf{f}}_\ell = \mathbf{k}_\ell^{-1}\mathbf{Y}_{\varepsilon,\ell}1_{\{|\mathbf{Y}_{\varepsilon,\ell}|>\mathrm{Sig}_\ell\}}, \quad \ell \leq L\,.$$

# Convergence results

- Let the Sobolev ball with radius $M$ be
$$\mathcal{W}^s(M) = \left\{ \mathbf{f} \in \mathbb{L}^2(\mathbb{S}^2), \sum_{\ell \geq 0} \ell^{2s} |\mathbf{f}_\ell|^2 \leq M^2 \right\}.$$
- Let the set of kernels with DIP $\nu$ be defined as
$$\mathcal{K}_\nu(Q) = \left\{ \mathbf{k} \in \mathbb{L}^2(\mathbb{T}), |\mathbf{k}_\ell^{-1}| \leq Q \ell^\nu \right\}$$

.

## Theorem

*Let* $L \sim \varepsilon^{-1}$.

$$\sup_{\substack{\mathbf{f} \in \mathcal{W}^s(M) \\ \mathbf{K} \in \mathcal{K}_\nu(Q)}} \mathbb{E} \|\tilde{\mathbf{f}} - \mathbf{f}\| \leq C \left( \varepsilon \sqrt{|\log \varepsilon|} \right)^{\frac{2s}{2s+2\nu+1}}.$$

*Those rates are minimax optimal up to logarithmic factors.*

# Semi-blind deconvolution framework

The convoluting kernel $\mathbf{k}$ is itself subject to experimental incertitude.

## Modelization

We have access to $\mathbf{k}_\delta = \mathbf{k} + \delta \mathbf{B}$, where $\mathbf{B}$ is a GWN on $\mathbb{L}^2(\mathbb{T})$.

## Goal

Recover $\mathbf{f}$ from

$$\begin{cases} \mathbf{Y}_\varepsilon & = & \mathbf{k} * \mathbf{f} + \varepsilon \mathbf{W} \\ \mathbf{k}_\delta & = & \mathbf{k} + \delta \mathbf{B} \end{cases},$$

while controlling the error due to $\varepsilon \mathbf{W}$ and $\delta \mathbf{B}$.

## Projection on harmonic functions

$$\begin{cases} \mathbf{Y}_{\varepsilon,\ell} & = & \mathbf{k}_\ell \mathbf{f}_\ell + \varepsilon \mathbf{W}_\ell \\ \mathbf{k}_{\delta,\ell} & = & \mathbf{k}_\ell + \delta \mathbf{B}_\ell \end{cases}$$

where $\mathbf{W}_\ell$ and $\mathbf{B}_\ell$ are i.i.d. are $\mathcal{N}(0, 1)$.

# Matricial representation

**Signal** : $\mathbf{Y} = \mathbf{k} * \mathbf{f} + \varepsilon \mathbf{W}$

$$\begin{pmatrix} \mathbf{Y}_{\varepsilon,1} \\ \mathbf{Y}_{\varepsilon,2} \\ \vdots \\ \mathbf{Y}_{\varepsilon,\ell} \end{pmatrix} = \begin{pmatrix} \mathbf{k}_1 & & & \\ & \mathbf{k}_2 & & \\ & & \ddots & \\ & & & \mathbf{k}_\ell \end{pmatrix} \begin{pmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \vdots \\ \mathbf{f}_\ell \end{pmatrix} + \varepsilon \begin{pmatrix} \mathbf{W}_1 \\ \mathbf{W}_2 \\ \vdots \\ \mathbf{W}_\ell \end{pmatrix}, \quad \begin{pmatrix} \mathbf{W}_1 \\ \mathbf{W}_2 \\ \vdots \\ \mathbf{W}_\ell \end{pmatrix} \sim \mathcal{N}(0, \mathbf{I}_\ell)$$

**Kernel** : $\mathbf{k}_\delta = \mathbf{k} + \delta \mathbf{B}$

$$\begin{pmatrix} \mathbf{k}_{\delta,1} \\ \mathbf{k}_{\delta,2} \\ \vdots \\ \mathbf{k}_{\delta,\ell} \end{pmatrix} = \begin{pmatrix} \mathbf{k}_1 \\ \mathbf{k}_2 \\ \ddots \\ \mathbf{k}_\ell \end{pmatrix} + \delta \begin{pmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \\ \ddots \\ \mathbf{B}_\ell \end{pmatrix}, \quad \begin{pmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \\ \vdots \\ \mathbf{B}_\ell \end{pmatrix} \sim \mathcal{N}(0, \mathbf{I}_\ell)$$

# Final Algorithm

**Reconstruction formula** :

$$\mathbf{f} = \sum_{\ell \geq 0} \mathbf{k}_\ell^{-1} \left(\mathbf{k} * \mathbf{f}\right)_\ell u_\ell \,.$$

For $\ell \leq L$, define the thresholded versions

$$\underline{\text{Signal}} : \tilde{\mathbf{Y}}_{\varepsilon,\ell} = \mathbf{Y}_{\varepsilon,\ell} 1_{\left\{|\mathbf{Y}_{\varepsilon,\ell}| > \tau\varepsilon\sqrt{|\log \varepsilon|}\right\}} ,$$

$$\underline{\text{Kernel}} : \tilde{\mathbf{k}}_{\delta,\ell}^{-1} = \mathbf{k}_{\delta,\ell}^{-1} 1_{\left\{|\mathbf{k}_{\delta,\ell}|^{-1} < \left(\kappa\delta\sqrt{|\log \delta|}\right)^{-1}\right\}} ,$$

and

$$\tilde{\mathbf{f}}_\ell = \tilde{\mathbf{k}}_{\delta,\ell}^{-1} \tilde{\mathbf{Y}}_{\varepsilon,\ell} \,.$$

# Rates of convergence

**Theorem (Delattre, Hoffmann, Picard, and V. (2012))**

*Let* $L \sim \delta^{-1/\nu} \wedge \varepsilon^{-2}$.

$$\sup_{\substack{\mathbf{f} \in \mathcal{W}^s(M) \\ \mathbf{k} \in \mathcal{K}_\nu(Q)}} \mathbb{E}\|\mathbf{f} - \tilde{\mathbf{f}}\|_{\mathbb{L}^2(\mathbb{T})} \lesssim \left(\delta\sqrt{|\log\delta|}\right)^{1 \wedge \frac{s}{\nu}} \vee \left(\varepsilon\sqrt{|\log\varepsilon|}\right)^{\frac{2s}{2s+2\nu+1}}.$$

*Furthermore, the estimator is minimax optimal.*

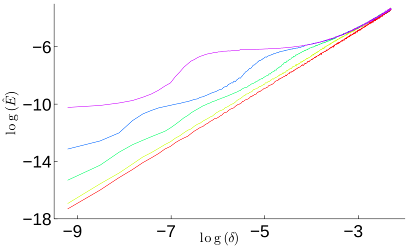



FIGURE : Estimation of the rate exponent when $\varepsilon \ll \delta$. Bottom-to-top : $s/\nu = 4.5, 1.1, 0.9, 0.7, 0.6$.

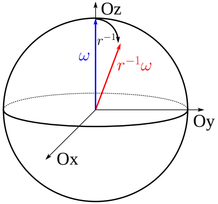
# Introduction

Problem introduced and solved by Healy, Hendriks, and Kim (1998) and Kim and Koo (2002) (non adaptive).

Applications in astrophysics, brain shape modelling.

For $\mathbf{f} \in \mathbb{L}^2(\mathbb{S}^2)$ and $\mathbf{\Gamma} \in \mathbb{L}^2(\mathrm{SO}(3))$, let

$$\mathbf{Kf}(\omega) = \int_{r \in \mathrm{SO}(3)} \mathbf{\Gamma}(r)\mathbf{f}(r^{-1}\omega)dr\,.$$



# Example

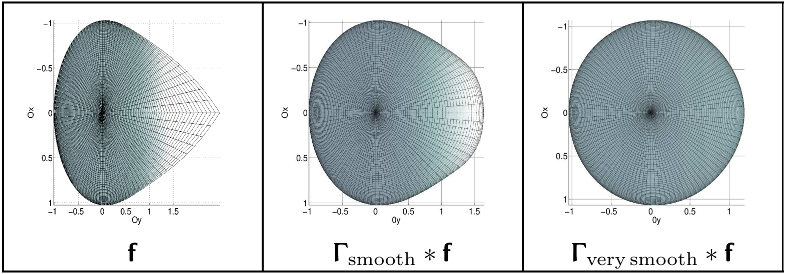


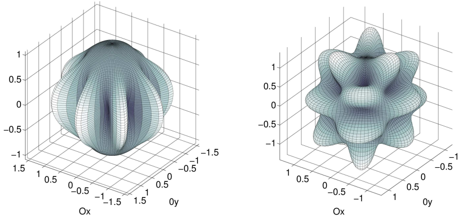

FIGURE : 'View from above' of $\mathbf{f}$ (left) and of $\Gamma * \mathbf{f}$ for different kernels $\Gamma$.

# Spherical harmonics

## Definition

For $\phi \in [0, 2\pi[, \theta \in [0, \pi[$, the $\ell, m$ spherical harmonic is

$$Z_{\ell,m}(\phi,\theta) = (-1)^m \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}} P_{\ell,m}(\cos\theta) e^{im\phi}, \; \ell \geq 0, \; -\ell \leq m \leq \ell \, .$$




(a) $Z_{8,-8}$.   (b) $Z_{8,4}$.

# Blockwise SVD

**Proposition**

*Let* $\mathbb{H}_\ell = \mathrm{span}\{Z_{\ell,m}, -\ell \leq m \leq \ell\}$*. Then* $\dim \mathbb{H}_\ell = 2\ell + 1$ *and we have*

$$\mathbf{K}(\mathbb{H}_\ell) \subset \mathbb{H}_\ell$$

*i.e.* $\mathbf{K}$ *admits a blockwise-SVD with respect to the spaces* $\mathbb{H}_\ell$.

Note $\overrightarrow{\mathbf{f}_\ell}$ the vector

$$\overrightarrow{\mathbf{f}_\ell} = \left(\langle \mathbf{f}, Z_{\ell,m}\rangle\right)_{-\ell\leq m\leq\ell}$$

and $\mathbf{\Gamma}_\ell$ the matrix

$$\mathbf{\Gamma}_\ell = \left[\langle \mathbf{\Gamma} * Z_{\ell,m}, Z_{\ell,n}\rangle\right]_{-\ell\leq m,n\leq\ell}.$$

Then

$$\overrightarrow{\mathbf{Kf}_\ell} = \mathbf{\Gamma}_\ell \overrightarrow{\mathbf{f}_\ell}$$

# The noisy model

Suppose now that we observe

$$\begin{cases} \mathbf{Y}_\varepsilon &= \mathbf{\Gamma} * \mathbf{f} + \varepsilon \mathbf{W} \\ \mathbf{\Gamma}_\delta &= \mathbf{\Gamma} + \delta \mathbf{B} \end{cases}$$

where $\mathbf{W}$ is a GWN on $\mathbb{L}^2(\mathbb{S}^2)$ and $\mathbf{B}$ is a GWN on $\mathbb{L}^2(\mathrm{SO}(3))$. Projecting onto each subspace $\mathbb{H}_\ell$, leads to the 'blockwise' sequential model :

$$\forall \ell \geq 0, \quad \begin{cases} \overrightarrow{\mathbf{Y}_{\varepsilon,\ell}} &= \mathbf{\Gamma}_\ell \overrightarrow{\mathbf{f}_\ell} + \varepsilon \overrightarrow{\mathbf{W}_\ell} \\ \mathbf{\Gamma}_{\delta,\ell} &= \mathbf{\Gamma}_\ell + \delta \mathbf{B}_\ell \end{cases}$$

where $\overrightarrow{\mathbf{W}_\ell} \in \mathbb{R}^{2\ell+1}$ and $\mathbf{B}_\ell \in M_{2\ell+1}(\mathbb{R})$ are constituted of independent Gaussian random variables.

# Matricial representation

**Signal** : $\mathbf{Y}_\varepsilon = \Gamma * \mathbf{f} + \varepsilon \mathbf{W}$

$$\begin{pmatrix} \overrightarrow{\mathbf{Y}_{\varepsilon,1}} \\ \vdots \\ \overrightarrow{\mathbf{Y}_{\varepsilon,\ell}} \end{pmatrix} = \begin{pmatrix} \Gamma_1 & & \\ & \ddots & \\ & & \Gamma_\ell \end{pmatrix} \begin{pmatrix} \overrightarrow{\mathbf{f}_1} \\ \vdots \\ \overrightarrow{\mathbf{f}_\ell} \end{pmatrix} + \varepsilon \begin{pmatrix} \overrightarrow{\mathbf{w}_1} \\ \vdots \\ \overrightarrow{\mathbf{w}_\ell} \end{pmatrix}$$

**Kernel** : $\Gamma_\delta = \Gamma + \delta \mathbf{B}$

$$\Gamma_{\delta,\ell} = \Gamma_\ell + \delta \, \mathbf{B}_\ell$$

# The adapted algorithm

**Reconstruction formula** :

$$\overrightarrow{\mathbf{f}_\ell} = \Gamma_\ell^{-1}\overrightarrow{\mathbf{Kf}_\ell}$$

Define the two thresholded levels :

$$\mathrm{Op}_\ell = \kappa\sqrt{2\ell+1}\delta\sqrt{|\log\delta|} \quad \text{and} \quad \mathrm{Sig}_\ell = \tau\sqrt{2\ell+1}\varepsilon\sqrt{|\log\varepsilon|}.$$

Then

$$\overrightarrow{\tilde{\mathbf{f}}_\ell} = \Gamma_{\delta,\ell}^{-1}\overrightarrow{\mathbf{Y}_{\varepsilon,\ell}} 1_{\{\|\Gamma_{\delta,\ell}^{-1}\|_{op}<\mathrm{Op}_\ell^{-1}\}} 1_{\left\{\|\overrightarrow{\mathbf{Y}_{\varepsilon,\ell}}\|>\mathrm{Sig}_\ell\right\}}, \quad \ell \leq L.$$

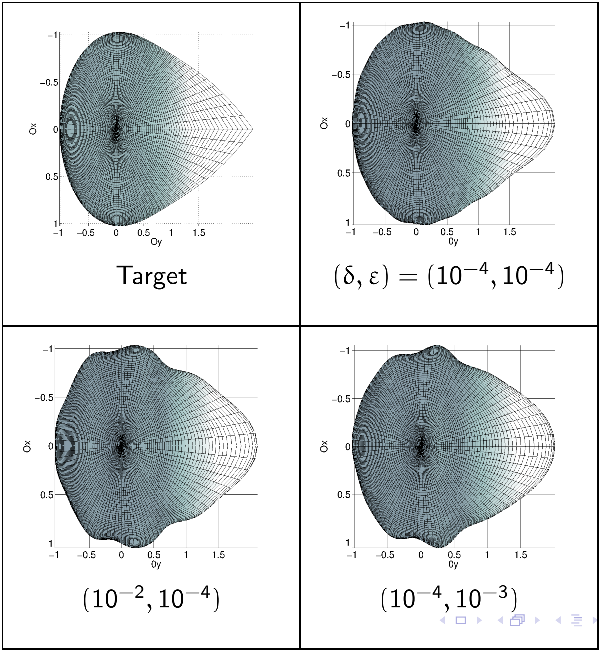
# Convergence results

- Let the Sobolev ball with radius $M$ be
$$\mathcal{W}^s(M) = \left\{\mathbf{f} \in \mathbb{L}^2(\mathbb{S}^2),\ \sum_{\ell \geq 0} \ell^{2s} \|\overrightarrow{\mathbf{f}_\ell}\|^2 \leq M^2\right\}.$$

- Let the set of kernels with DIP $\nu$ be defined as
$$\mathcal{K}_\nu(Q_1, Q_2) = \left\{\Gamma \in \mathbb{L}^2(\mathrm{SO}(3)),\ \|\Gamma_\ell^{-1}\|_{op} \leq Q_1 \ell^\nu \text{ and } \|\Gamma_\ell\|_{op} \leq Q_2 \ell^{-\nu}\right\}$$
.

**Theorem (Delattre, Hoffmann, Picard, and V. (2012))**

*Let $L \sim \varepsilon^{-1} \wedge \delta^{-2}$.*
$$\sup_{\substack{\mathbf{f} \in \mathcal{W}^s(M) \\ K \in \mathcal{K}_\nu(Q_1, Q_2)}} \mathbb{E}\|\tilde{\mathbf{f}} - \mathbf{f}\|_{\mathbb{L}^2(\mathbb{S}^2)} \leq C\left(\delta\sqrt{|\log \delta|}\right)^{1 \wedge \frac{2s}{2\nu+1}} \vee \left(\varepsilon\sqrt{|\log \varepsilon|}\right)^{\frac{2s}{2s+2\nu+2}}.$$
*Those rates are minimax optimal up to logarithmic factors.*


Ox
Oy
Target
$(\delta, \varepsilon) = (10^{-4}, 10^{-4})$
$(10^{-2}, 10^{-4})$
$(10^{-4}, 10^{-3})$

Delattre, S., M. Hoffmann, D. Picard, and T. V. (2012). Blockwise SVD with error in the operator and application to blind deconvolution. *EJS 6*, 2274–2308.

Healy, D., H. Hendriks, and P. Kim (1998). Spherical deconvolution. *J. Multivariate Anal. 67*, 1–22.

Kim, P. and Y. Koo (2002). Optimal spherical deconvolution. *J. Multivariate Anal. 80*, 21–42.