Stuck Walks: a conjecture of Erschler, Tóth and Werner

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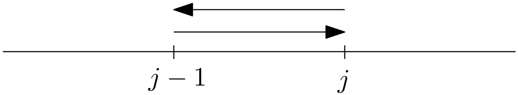
Definition

$X := (X_n)_{n\geq 0}$: nearest neighbor walk on the integer lattice $\mathbb{Z}$, starting at 0, i.e. $X_0 = 0$.
Local time, at time k, on the non-oriented **edge** $\{j-1, j\}$:

$$l_k(j) \quad = \quad \sum_{m=1}^{k} \mathbb{1}_{\{\{X_{m-1}, X_m\} = \{j-1, j\}\}}.$$



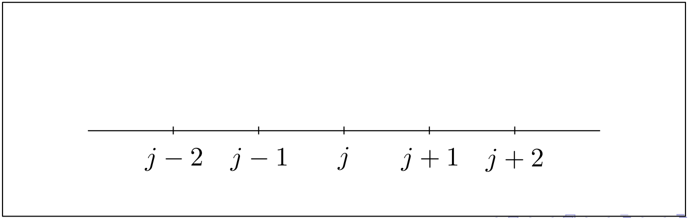
Definition

Fix $\alpha \in \mathbb{R}$. Define the **local stream**:

$$\Delta_k(j) = -\alpha l_k(j-1) + l_k(j) - l_k(j+1) + \alpha l_k(j+2).$$

Conditional transition probability:

$$\mathbb{P}(X_{k+1} = X_k \pm 1 | \mathcal{F}_k) \quad = \quad \frac{e^{\pm\beta\Delta_k(X_k)}}{e^{-\beta\Delta_k(X_k)} + e^{\beta\Delta_k(X_k)}}, \text{ where } \beta > 0.$$



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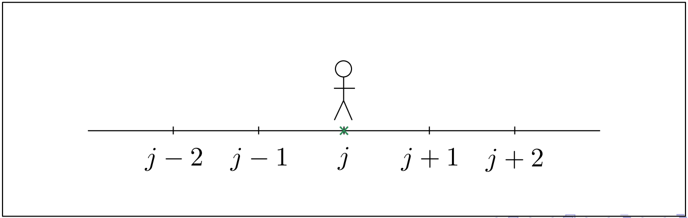

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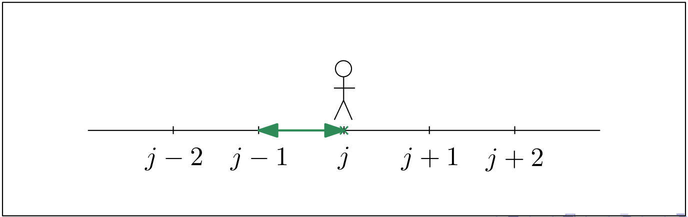

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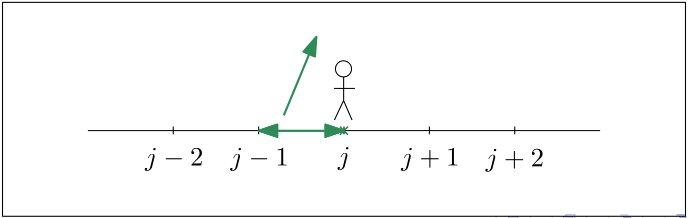

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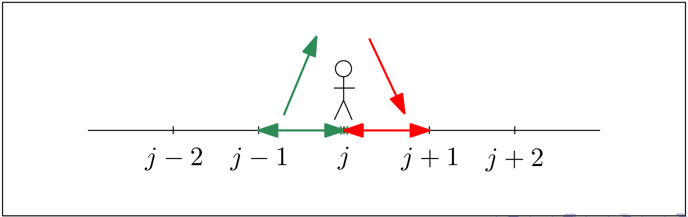

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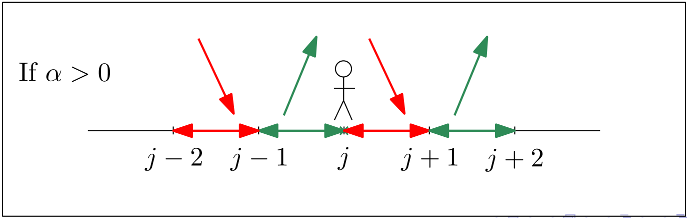

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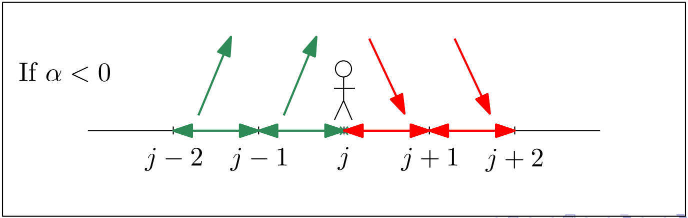

Generalization of the true self-repelling walk (TSRW) in one dimension.

- $\alpha = 0$: TSRW with edge repulsion, non-degenerate scaling limits for $X_k/k^{2/3}$ proved by Tóth (1995).
- $\alpha \in [-1, 1/3[$: similar scaling behavior expected. Remark: $\alpha = -1$: TSRW with site repulsion.
- $\alpha = 1/3$: $k^{2/5}$?
- $\alpha \in (-\infty, -1)$: trapping environments are self-built, slow phase.
- $\alpha > 1/3$: Stuck Walks: competition between repulsion and attraction, localization of the walk on an arbitrarily large interval depending on α.
- Reference: *Some locally self-interacting walks on the integers* (2010), by Erschler, Tóth and Werner.

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Notations

Define $\alpha_1 = +\infty$ and for all $L \geq 2$:

$$\alpha_L = \frac{1}{1 + 2\cos(\frac{2\pi}{L+2})},$$

so that $(\alpha_L)_{L\geq 1}$ decreases from $+\infty$ to $1/3$.
Let R' be the set of points that are visited **infinitely often**, i.e.

$$R' \quad = \quad \{j \in \mathbb{Z} : l_\infty(j) + l_\infty(j+1) = \infty\}.$$

Stuck Walks (2010)

Theorem (Erschler, Tóth and Werner, 2010)

Suppose that $L \geq 1$. We have:
- If $\alpha < \alpha_L$, then, almost surely, $|R'| \geq L+2$, or $R' = \emptyset$;
- If $\alpha \in (\alpha_{L+1}, \alpha_L)$, then the probability that $|R'| = L+2$ is positive.

Conjecture (Erschler, Tóth and Werner, 2010)

If $\alpha \in (\alpha_{L+1}, \alpha_L)$, then $|R'| = L+2$ almost surely.

Results

Theorem (K, 2013)

Fix $L \geq 1$ *and assume that* $\alpha \in (\alpha_{L+1}, \alpha_L)$*, then the walk localizes on* $L+2$ *or* $L+3$ *sites almost surely, i.e.* $|R'| \in \{L+2, L+3\}$ *a.s.*

Theorem (K, 2013)

Assume that that $\alpha \in (1, +\infty) = (\alpha_2, \alpha_1)$*, then the walk localizes on* 3 *sites almost surely, i.e.* $|R'| = 3$ *a.s.*

Sketch of the proof - Finite Range

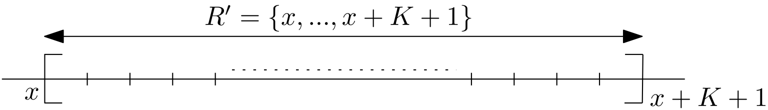
Fix $L \geq 1$ and $\alpha \in (\alpha_{L+1}, \alpha_L)$.

Proposition (K, 2013)

The walk almost surely visits finitely many sites.

- Discards the transience.
- From $\mathtt{ETW}$, $|R'| \geq L + 2$ a.s.

Therefore, sufficient to prove $|R'| \leq L + 3$ a.s.



$$\text{For any } j \in \{1, ..., K\} : d_j \sim \frac{\Delta_k(x+j)}{k} \to 0$$

$$\text{For any } j \in \{0, ..., K+2\} : l_j \sim \frac{l_k(x+j)}{k} \to ?$$

Solutions $(l_0, ..., l_{K+2})$ of the linear system defined by:

$$d_1 = d_2 = ... = d_K = 0,\ l_0 = l_{K+2} = 0 \text{ and } \sum_{j=1}^{K+1} l_j = 1,$$

where, for all $j \in \{1, ..., K\}$,

$$d_j = -\alpha l_{j-1} + l_j - l_{j+1} + \alpha l_{j+2}.$$

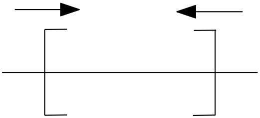
Define $d_0 = -l_1 + \alpha l_2$ and $d_{K+1} = -\alpha l_K + l_{K+1}$.

The linear system

- If $K < L$

$$\frac{\Delta_k(x)}{k} \sim d_0 < 0 \qquad\qquad 0 < d_{K+1} \sim \frac{\Delta_k(x+K+1)}{k}$$

- If $K = L$ or $L + 1$



$$\frac{\Delta_k(x)}{k} \sim d_0 > 0 \qquad\qquad 0 > d_{K+1} \sim \frac{\Delta_k(x+K+1)}{k}$$

- $K > L + 1$: Get more results in order to emphasize instability of some vertices through the **Rubin's construction**.

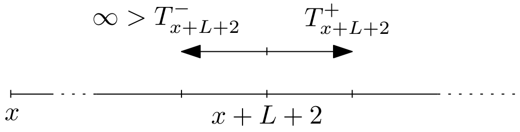
Rubin's construction

First used by Davis (1990) and Sellke (1994). Tarrès (2011) introduced a **variant** $\Rightarrow$ **powerful couplings** for VRRW.

We **generalize** it to a **larger class** of processes in order to get the conclusion.

Idea: define a continuous-time walk $(\widetilde{X}_t)_t$ on $\mathbb{Z}$ in order to couple to with $(X_k)_k$.

$$T_y^{\pm} := \sum_{k \geq 0} (\text{Clock attached to } (y, y \pm 1) \text{ when jump } y \pm 1 \to y)\,.$$



Proposition

Almost surely

$$\left\{x = \inf R'\right\} \subset \{Z_\infty(x + L + 3) < \infty\} .$$

THANK YOU!

Similar results - References

- *Stuck Walks*, 2010, A. Erschler, B. Tóth, W. Werner.
- *Some locally self-interacting walks on the integers*, 2010, by Erschler, Tóth and Werner.
- *Vertex-reinforced random walk on $\mathbb{Z}$ eventually gets stuck on five points*, 2004, P. Tarrès.
- *Localization of reinforced random walks*, 2011, P. Tarrès.
- *Localization on 4 sites for vertex-reinforced random walks on $\mathbb{Z}$*, 2012, A-L. Basdevant, B. Schapira, A. Singh.
- *Localization of a vertex reinforced random walk on $\mathbb{Z}$ with sub-linear weight*, 2012, A-L. Basdevant, B. Schapira, A. Singh.