Reduction of complexity in an heterogeneous population with two timescales

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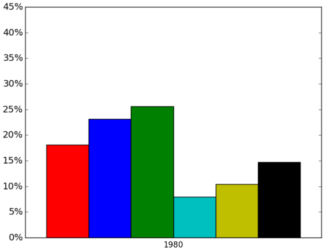
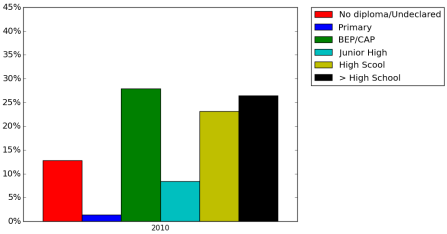
Plan

Motivations

Initial motivation: model and better understand human longevity.

- Longevity data are often aggregated data such as national data.
- Issue: strong heterogeneity in populations. Ex: Expected lifetime at old ages for individuals with high school diploma is 20% higher than for individuals with no diploma (INSEE).
- Social structure of populations is not stable and can change at fast pace.(Evolution of French diploma repartition for 30-45 years old)





Motivations

More "micro" description of the population needed to understand aggregated variables.

- Aggregation methods are important in several fields (economics, biology, operations research..)
- Goal : model complex link between the micro and macro.
- Separation of timescales between different types of phenomenons: permits to make some averaging approximations and thus to better understand the aggregation process.

Our framework:

Study of a stochastic model of population when "social changes" occur at a fast pace in comparison to the demographic time scale of a life time.

Plan

Population process

- We consider a population divided in p homogeneous subgroups: individuals in the same subgroup are indistinguishable and have the same "demographic" behavior.
- Population process: describes the evolution of the number of individuals in each subgroup. *càdlàg* process $Z = (Z^1, .., Z^p) \in \mathbb{N}^p$, adapted to a given filtration $(\mathcal{F}_t)_{t\geq 0}$ where Z^i_t is the number of individual in subgroup i at time t.
- Total size of the the population: $\bar{Z} =< Z, \mathbf{1} >= \sum Z^i$.

Assumption and remark

- Two kinds of events occur:
  - Demographic events: birth or death in one of the subgroups. $\Delta\bar{Z}_t = \pm 1$, $\Delta Z_t^i = \pm 1$.
  - Mixing events: An individual changes of characteristics and moves from a subgroup i to j. $\Delta\bar{Z}_t = 0$, $\Delta Z_t^j = 1$,$\Delta Z_t^i = -1$.
- Assumption: No events happen simultaneously.
- (Z_t^i) only increase/decrease of one when an event happens in the subpopulation i. But Z is *not a multivariate birth and death process* since components have simultaneous jumps when mixing events happen.

Notations for the set of all possible events

- Canonical basis in $\mathbb{R}^p$: $(\mathbf{e}_i)_{1\leq i\leq p}$. $\mathbf{e}_\infty = \mathbf{0}$.
- Mixing events: natural notation (i,j) for mixing event from i to j.

  Set of all mixing events: $\mathcal{I}^{mix} = \{(i,j) \in [\![1,p]\!]^2;\ i \neq j\}$.
- Demographic events: we define the "metaphysical" infinite population $\{\infty\}$ of individuals not born yet and who died. Notation (∞, i) $((i,\infty))$ for the event birth (death) in subgroup i.

  Set of demographic event: $\mathcal{I}^{dem} = ([\![1,p]\!] \times \{\infty\}) \cup (\{\infty\} \times [\![1,p]\!])$.
- Set of all events: $\mathcal{I} = \mathcal{I}^{dem} \cup \mathcal{I}^{mix}$.
- Jump associated to event of type (i,j): $\phi(i,j) = \mathbf{e}_j - \mathbf{e}_i$.

$$\{(i,j) \text{ happens at time } t\} = \{\Delta Z_t = \phi(i,j)\}.$$

Generating counting processes

Definition

For all $(i,j) \in \mathcal{I}$, we denote by $\mathbf{N} = (N^{ij})_{(i,j)\in\mathcal{I}}$ the a multivariate counting process with:

$$N_t^{ij} = \sum_{s \leq t} \mathbb{1}_{\{\Delta Z_s = \phi(i,j)\}} \quad t \geq 0, \tag{1}$$

Counting processes easily characterized by their intensity. Gives martingale property and makes computations very simple.

Markov framework

Assumption (Intensity of the counting processes)

For all $(i,j) \in \mathcal{I}$ the counting process N^{ij} has the $\{\mathcal{F}_t\}$-intensity $q^{ij}(Z_s)$:

$M_t^{ij} = N_t^{ij} - \int_0^t q^{ij}(Z_s)ds$ *is a $\{\mathcal{F}_t\}$-local martingale.*

We assume that there are no explosions of the processes.

$\mathbf{N}$ is a strong Markov process.

Link between the counting processes and the population process

Functionals of population process can be written as sum of their jumps:

$$f(Z_t) = f(Z_0) + \sum_{s \leq t} (f(Z_{s^-} + \Delta Z_s) - f(Z_{s^-}))\mathbf{1}_{\{\Delta Z_s \neq 0\}}$$

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Martingale problem

$$f(Z_t) - f(Z_0) - \int_0^t \left(\mathcal{K}^{mix} + \mathcal{K}^{dem}\right) f(Z_s)ds \quad \text{is a martingale, with}$$

$$\mathcal{K}^{mix\ (dem)} f(\mathbf{z}) = \sum_{i,j\in\mathcal{I}^{mix\ (dem)}} q^{ij}(\mathbf{z})\left(f(\mathbf{z} + \phi(i,j)) - f(\mathbf{z})\right)$$

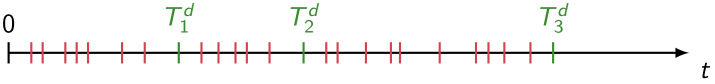
Z is a continuous time Markov chain.

Plan

Instantaneous mixing with respect to demographic timescale

Separation of timescale

- Let's define the *nth* demographic event T_n^d:
  $T_n^d = \inf\{t > T_{n-1}^d ; \Delta \bar{Z}_t \neq 0\}$ $(T_0 = 0)$.
- Hypothesis: mixing intensities $>>$ demographic intensities.



Idea:

- Decouple mixing and demographic events by isolating "mixing excursions" which evolve on finite spaces between two demographic events.
- Process reinterpreted as a Markov chain of killed trajectories, reborn after a demographic event.

Mixing excursions

- Time between two demographic events: $D_n = T_{n+1}^d - T_n^d$
- **nth mixing excursion of the population process**; process after nth demographic event and killed at the $(n+1)th$:

$$\mathcal{E}_t^n = \begin{cases} Z_{t+T_n^d} & \text{if } t < D_n \\ \partial & \text{if } t \geq D_n. \end{cases} \tag{3}$$

Population process is equivalent to the sequence of Mixing excursions $(\mathcal{E}^n)_{n\in\mathbb{N}}$:

$$Z_t = \sum_{n\geq 0} \mathbb{1}_{\{T_n^d \leq t\}} \mathcal{E}_{t-T_n^d}^n \quad \forall t \geq 0. \tag{4}$$

Population before the first demographic event

Population killed at the first demographic event:

- Only mixing events happen on $\{t < T_1^d\}$

$$f(Z_t) = f(Z_0) + \sum_{(i,j)\in\mathcal{I}^{mix}} \int_0^t \left(f(Z_{s^-} + \phi(i,j)) - f(Z_{s^-})\right) dN_s^{ij}.$$

- T_1^d is the first jump time of $\bar{N}^{dem} = \sum_{(i,j)\in\mathcal{I}^{dem}} N^{ij}$.
- Happens with intensity $q^{dem}(Z_t)\mathbb{1}_{\{t<T_1^d\}}$ with

$$q^{dem}(\cdot) = \sum_{i=1}^{p} q^{\infty i}(\cdot) + q^{i\infty}(\cdot).$$

Mixing excursions are "pure mixing" processes, killed at a rate , of generator:

$$\mathcal{K}^{\partial} f(z) = \mathcal{K}^{mix} f(z) - q^{dem}(z) f(z)$$

.

Killed pure mixing process

Pure mixing process

- $X = (X^x)_{x \in \mathbb{N}^p}$ processes with semigroup P^{mix} generated by $\mathcal{K}^{mix}$.
- Conservative process: for all initial population $z \in \mathbb{N}^p$, X^z has value in the finite space $\mathcal{U}_{\bar{z}}$ of populations of size $\bar{z} =< z, 1 >$:

$$\mathcal{U}_{\bar{z}} = \{x \in N^p \ \bar{z} = \bar{x}\}.$$

Subordinated process

- The semigroup P^{∂} is subordinated to the pure mixing process,i.e:

$$P_s^{\partial} f \leq P_s^{mix} f \quad \forall f : \mathbb{N}^p \to \mathbb{R}^+, \ s \geq 0.$$

- P^{∂} can be realized by the expectation of a pure mixing process discounted by a multiplicative functional of the process. Here:

$$P_s^{\partial} f(z) = \mathbb{E}^X [f(X_s^z) e^{-\int_0^s q^{dem}(X_u^z) du}]. \qquad (5)$$

Fast mixing process

Hypothesis: mixing intensities >> demographic intensities.

- The population depends on a small parameter ϵ. The two timescale population process Z^ϵ has generator $\frac{1}{\epsilon}\mathcal{K}^{mix} + \mathcal{K}^{dem}$.
- Killed "fast" mixing representation:

$$P_s^{\partial,\epsilon} f(z) = \mathbb{E}^{X^\epsilon}[f(X_s^{z,\epsilon})\exp(-\int_0^s q^{dem}(X_u^{z,\epsilon})du)]$$

with X^ϵ some "fast" pure mixing process of generator $\frac{1}{\epsilon}\mathcal{K}^{mix}$.

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- Realization on the same probability space: $X_\tau^{x,\epsilon} = X_{\frac{\tau}{\epsilon}}^x \quad \forall \tau \geq 0$.

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- Killed "fast" mixing process with changed time pure mixing:

$$P_s^{\partial,\epsilon} f(z) = \mathbb{E}^X[f(X_{\frac{s}{\epsilon}}^z) \exp\left(-\int_0^s q^{dem}(X_{\frac{u}{\epsilon}}^z) du\right)] \quad (6)$$

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Ergodicity assumptions

Pure mixing defined on disjoint equivalence classes $(\mathcal{U}_n)_{n\in\mathbb{N}}$.

Assumption

$\forall n \in \mathbb{N}$, the pure mixing process restricted to $\mathcal{U}_n$ is strongly irreducible. Thus the mixing process restricted to $\mathcal{U}_n$ admits a unique stationary measure ν_n. In particular, $\forall z \in \mathbb{N}^p$, and bounded functions f:

$$\frac{1}{t}\int_0^t f(X_s^z)ds \underset{t\to+\infty}{\longrightarrow} \nu_{\bar{z}}(f) = \int_{\mathcal{U}_{\bar{z}}} f(x)\nu_{\bar{z}}(dx) \quad a.s,$$

$$\mathbb{E}^X[f(X_t^z)] \underset{t\to+\infty}{\longrightarrow} \nu_{\bar{z}}(f).$$

$$P_s^{\partial,\epsilon} f(z) = \mathbb{E}^X[f(X_{\frac{s}{\epsilon}}^z)e^{-\epsilon\int_0^{\frac{s}{\epsilon}} q^{dem}(X_u^z)du}] \underset{\epsilon\to 0}{\longrightarrow} \nu_{\bar{z}}(f)e^{-s\nu_{\bar{z}}(q^{dem})}.$$

In particular, the distribution of the first demographic event tends to an exponential variable of intensity $\nu_{\bar{z}_0}(q^{dem})$.

Averaged aggregated process

- Aggregated process $\bar{Z}^{\epsilon} =< Z^{\epsilon}, \mathbb{1} >$ is not a Markov process (intensities depend on the whole structure of the population, and are not constant between two events due to mixing events)
- Birth and death intensities:

$$q^b(\cdot) = \sum_{i=1}^{p} q^{(\infty,i)}(\cdot) \quad \text{and} \quad q^d(\cdot) = \sum_{i=1}^{p} q^{(i,\infty)}(\cdot)$$

- When timescale of mixing process becomes instantaneous: aggregated process converges to an "averaged" markovian process.

Theorem (Convergence of the aggregated process)

Let $\bar{Z}^{\bar{z},\epsilon}$ be the aggregated process starting with an initial size of $\bar{z}$. $\forall \bar{z} \in N$, $\bar{Z}^{\bar{z},\epsilon}$ converge in distribution towards a Birth and Death process $\bar{\mathcal{Z}}^{\bar{z}}$ with birth and death intensities defined respectively by:

$$Q(\bar{z}, \bar{z}+1) = \nu_{\bar{z}}(q^b) \quad Q(\bar{z}, \bar{z}-1) = \nu_{\bar{z}}(q^d) \quad \forall \bar{z} \in \mathbb{N}.$$

Conclusion and perspectives

- Dynamic of the aggregated process can be approximated by a simpler Markovian process.
- The aggregated birth and death intensities are "averaged" with respect to the stationary distribution of the mixing process.
- Perspective: Understand the link between this limit and an "intermediate" limit given by quasi stationary distributions.

Thank You For Your Attention