

Weak approximation of martingale representations

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Martingale representations

- $(\Omega, \mathcal{F}, \mathbb{P})$ probability space
- W d -dimensional Brownian motion, (\mathcal{F}_t^W) its (\mathbb{P} -completed) natural filtration
- $T > 0$, $\forall H \in L^2(\mathcal{F}_T^W)$, $\exists \phi$ (\mathcal{F}_t^W) -predictable such that :

$$H = \mathbb{E}[H] + \int_0^T \phi \cdot dW$$

- Problem : ϕ non-explicit
- Motivations : ϕ represents hedging strategy in mathematical finance

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Notations

- $T > 0$, $D([0, T], \mathbb{R}^d)$ càdlàg paths with values in \mathbb{R}^d equipped with the supremum norm :

$$\|\omega\|_\infty = \sup\{|\omega(t)|, t \in [0, T]\}$$

- For $\omega \in D([0, T], \mathbb{R}^d)$, we distinguish :
 - $\omega(t)$ value of ω at t
 - $\omega_t = \omega(t \wedge \cdot) \in D([0, T], \mathbb{R}^d)$ path of ω stopped at t
- Similar for stochastic processes

Framework

- $dX(t) = b(t, X_t)dt + \sigma(t, X_t)dW(t)$, $X_0 = x_0 \in \mathbb{R}^d$
- $W \xrightarrow{\text{Ito map}} X \xrightarrow{g} H = g(X_T) \in L^2(\mathcal{F}_T^W)$
 - X non-Markovian
 - No differentiability assumptions on b or σ
 - No differentiability assumptions on g
- $H = \mathbb{E}[H] + \int_0^T \phi \cdot dW$
- Approximate ϕ by $(Z_n)_n$ **explicit** such that :

$$\left\| \int_0^T (Z_n(t) - \phi(t)) \cdot dW(t) \right\|_{\mathbb{L}^2} \leq C \sqrt{\frac{1 + \log n}{n}}$$

if g is Lipschitz continuous with respect to $\|\cdot\|_\infty$

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Non-anticipative functionals

- $F : (t, \omega) \in [0, T] \times D([0, T], \mathbb{R}^d) \mapsto F(t, \omega) \in \mathbb{R}$
non-anticipative if :

$$F(t, \omega) = F(t, \omega_t)$$

- $F : (\Lambda_T, d_\infty) \longrightarrow \mathbb{R}$ where :
 $\Lambda_T = \{(t, \omega(t \wedge \cdot)), (t, \omega) \in [0, T] \times D([0, T], \mathbb{R}^d)\}$ space of
stopped path equipped with :

$$d_\infty((t, \omega), (t', \omega')) = \|\omega_t - \omega'_{t'}\|_\infty + |t - t'|$$

Notions of differentiability for functionals

- A non-anticipative functional F is said to be :
 - horizontally differentiable at $(t, \omega) \in \Lambda_T$ if

$$DF(t, \omega) = \lim_{h \rightarrow 0^+} \frac{F(t+h, \omega_t) - F(t, \omega_t)}{h}$$

exists. DF is called the horizontal derivative of F .

- vertically differentiable at $(t, \omega) \in \Lambda_T$ if the map :

$$\begin{aligned} \mathbb{R}^d &\longrightarrow \mathbb{R} \\ e &\mapsto F(t, \omega_t + e \mathbf{1}_{[t, T]}) \end{aligned}$$

is differentiable at 0. Its gradient at 0 is called the vertical derivative of F at (t, ω) :

$$\nabla_\omega F(t, \omega) = (\partial_i F(t, \omega), i = 1, \dots, d) \in \mathbb{R}^d$$

with

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Functional Ito formula

Theorem (Dupire(09) & Cont, Fournié(13))

Let S be a continuous semimartingale defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and F a non-anticipative functional which is :

- horizontally differentiable
- twice vertically differentiable
- continuity properties of F , $\mathcal{D}F$, $\nabla_{\omega}F$ and ∇_{ω}^2F

we have, for any $t \in [0, T]$,

$$\begin{aligned}
 F(t, S_t) - F(0, S_0) &= \int_0^t \mathcal{D}F(u, S_u) du + \int_0^t \nabla_{\omega}F(u, S_u) \cdot dS(u) \\
 &\quad + \frac{1}{2} \int_0^t \text{tr}(\nabla_{\omega}^2F(u, S_u) d[S](u)) \quad \mathbb{P}\text{-a.s.}
 \end{aligned}$$

Framework

- $dX(t) = b(t, X_t)dt + \sigma(t, X_t)dW(t)$, $X(0) = x_0 \in \mathbb{R}^d$
- $b, \sigma : \Lambda_T \rightarrow \mathcal{M}_d(\mathbb{R})$ non-anticipative functionals

Assumption

$b, \sigma : (\Lambda_T, d_\infty) \rightarrow \mathcal{M}_d(\mathbb{R})$ are Lipschitz continuous

- $H = g(X_T)$ with g satisfying :

Assumption

g has polynomial growth with respect to $\|\cdot\|_\infty$

Conditional expectation as non-anticipative functional

- $Y(t) = \mathbb{E}[g(X_T)|\mathcal{F}_t^W] = F(t, W_t)$
- If $\exists F$ smooth, then by functional Ito formula,

$$g(X_T) = \mathbb{E}[g(X_T)] + \int_0^T \underbrace{\nabla_\omega F(t, W_t)}_{\phi(t)} \cdot dW(t)$$

- **Problem** : F is in general not smooth !
- **Idea** : Approximate X by ${}_n X$ such that :

$${}_n Y(t) = \mathbb{E}[g({}_n X_T)|\mathcal{F}_t^W] = F_n(t, W_t)$$

with F_n smooth

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Euler approximations

- $n \in \mathbb{N}$, $\delta = \frac{T}{n}$, grid $(t_j = j\delta, j = 0, \dots, n)$
- For $\omega \in D([0, T], \mathbb{R}^d)$, we define the piecewise constant Euler approximation of X along ω as follows :

Definition (Euler scheme)

${}_nX(\omega)$ is constant in each interval $[t_j, t_{j+1})$ with ${}_nX(0, \omega) = x_0$ and for $0 \leq j \leq n-1$,

$$\begin{aligned} {}_nX(t_{j+1}, \omega) &= {}_nX(t_j, \omega) + b(t_j, {}_nX_{t_j}(\omega))\delta \\ &\quad + \sigma(t_j, {}_nX_{t_j}(\omega))(\omega(t_{j+1}-) - \omega(t_j-)) \end{aligned}$$

- When $\omega = W$, ${}_nX(W_T)$ Euler scheme of X

Definition of the functional F_n

- $F_n(t, \omega_t) = \mathbb{E} \left[g(nX(\omega_t \oplus_t B)) \right]$ with B Brownian motion independent of W
- Concatenation of paths $\omega, \omega' \in D([0, T], \mathbb{R}^d)$ at $t \in [0, T]$:

$$\omega \oplus_t \omega' = \omega_t \oplus_t \omega' = \begin{cases} \omega(u) & u \in [0, t) \\ \omega(t) + \omega'(u) - \omega'(t) & u \in [t, T] \end{cases}$$

- By independence of increments of W ,
 $F_n(t, W_t) = \mathbb{E} [g(nX(W_T)) | \mathcal{F}_t^W]$

Main results

Theorem

The functional F_n defined by $F_n(t, \omega_t) = \mathbb{E} \left[g(nX(\omega_t \oplus_t B)) \right]$ satisfies all the properties to apply the functional Ito formula. And we have :

$$g(nX(W_T)) = \mathbb{E}[g(nX(W_T))] + \int_0^T \underbrace{\nabla_{\omega} F_n(t, W_t)}_{Z_n(t)} \cdot dW(t) \quad \mathbb{P} - a.s.$$

Main results

Corollary

If we assume in addition that g is Lipschitz continuous with respect to $\|\cdot\|_\infty$, we have :

$$\left\| \int_0^T (Z_n(t) - \phi(t)) \cdot dW(t) \right\|_{\mathbb{L}^2} \leq C \sqrt{\frac{1 + \log n}{n}}$$

Proof : $\left\| \sup_t |{}_n X(W_T)(t) - X(t) \right\|_{\mathbb{L}^2} \leq C \sqrt{\frac{1 + \log n}{n}}$

Sketch of proof

- ${}_nX(\omega)$ depends only on $\omega(t_1-) - \omega(0), \dots, \omega(t_n-) - \omega(t_{n-1}-)$
- Consequently, for $t_k \leq t < t_{k+1}$, $g({}_nX(\omega_t \oplus_t B))$ depends only on $B(t_{k+1}) - B(t), B(t_{k+2}) - B(t_{k+1}), \dots, B(t_n) - B(t_{n-1})$
- $g({}_nX(\omega_t \oplus_t B)) = f(B(t_{k+1}) - B(t), \dots, B(t_n) - B(t_{n-1}))$

Sketch of proof

- A jump of size z of ω at time t is equivalent to a "jump" of $B(t_{k+1}) - B(t)$:

$$g({}_n X((\omega_t + z \mathbf{1}_{[t, T]}) \oplus_t B)) = f(B(t_{k+1}) - B(t) + z, \dots, B(t_n) - B(t_{n-1}))$$

- Vertical differentiability \iff differentiability of $\mathbb{E}[f(\dots)]$ with respect to z , ensured by the regularity of the Gaussian density function
- For horizontal differentiability, we have similarly :

$$g({}_n X(\omega_t \oplus_{t+h} B)) = f(B(t_{k+1}) - B(t+h), \dots, B(t_n) - B(t_{n-1}))$$

for h small

Explicit vertical derivative

- $\nabla_{\omega} F_n(t, \omega_t) = \mathbb{E} \left[g(nX(\omega_t \oplus_t B)) \cdot \frac{B(t_{k+1}) - B(t)}{t_{k+1} - t} \right]$
 for $t_k \leq t < t_{k+1}$, $\forall \omega \in D([0, T], \mathbb{R}^d)$
- $\nabla_{\omega} F_n(t, W_t) = \mathbb{E} \left[g(nX(W_T)) \cdot \frac{W(t_{k+1}) - W(t)}{t_{k+1} - t} \middle| \mathcal{F}_t^W \right]$
 \rightsquigarrow BSDE

Remarks : Malliavin calculus

- $g({}_nX(W_T)) = h(W(t_1), W(t_2) - W(t_1), \dots, W(t_n) - W(t_{n-1}))$
- When h smooth, Clark-Ocone formula provides the same result
- When h not smooth, $g({}_nX(W_T))$ might not be Malliavin differentiable or its Malliavin derivative might not be explicit
- We require regularity conditions only on :

$$F_n(t, W_t) = \mathbb{E}[g({}_nX(W_T)) | \mathcal{F}_t^W]$$

which is more smooth than $g({}_nX(W_T))$

Remarks : BSDE

- $Z_n(t) = \mathbb{E} \left[g(nX(W_T)) \cdot \frac{W(t_{k+1}) - W(t)}{t_{k+1} - t} \middle| \mathcal{F}_t^W \right], t_k \leq t < t_{k+1}$
- In BSDE, $Z_n(t_k) = \mathbb{E} \left[Y_n(t_{k+1}) \cdot \frac{W(t_{k+1}) - W(t_k)}{t_{k+1} - t_k} \middle| \mathcal{F}_{t_k}^W \right]$
- Our case is a particular case of BSDE with $f \equiv 0$, the results of BSDE apply : (see Zhang(04))
 - $Z_n(t) = \mathbb{E} \left[\frac{1}{t_{k+1} - t} \int_t^{t_{k+1}} Z_n(s) ds \middle| \mathcal{F}_t^W \right]$
 - Z_n admits a càdlàg version
 - $\sum_{k=1}^n \mathbb{E} \int_{t_{k-1}}^{t_k} (|Z_t - Z_{t_{k-1}}|^2 + |Z_t - Z_{t_k}|^2) dt \leq \frac{C}{n}$

Remarks : BSDE

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Remarks : BSDE(cont'd)

- Decoupled FBSDE :

$$\begin{cases} dX(t) = b(t, X_t)dt + \sigma(t, X_t)dW(t), X(0) = x_0 \in \mathbb{R}^d \\ dY(t) = f(t, X_t, Y(t), Z(t))dt - Z(t)dW(t), Y(T) = g(X_T) \end{cases}$$

- X approximated by ${}_nX$ piecewise constant Euler scheme, $Y_n(t_k) = g({}_nX_T)$, for $1 \leq k \leq n$ and $t_{k-1} \leq t < t_k$,

$$\begin{cases} Y_n(t) = \mathbb{E} [Y_n(t_k) | \mathcal{F}_t^W] + f(t_{k-1}, {}_nX_{t_{k-1}}, Y_n(t_{k-1}), Z_n(t_{k-1}))(t_k - t) \\ Z_n(t) = \nabla_W (\mathbb{E} [Y_n(t_k) | \mathcal{F}_t^W]) \end{cases}$$

Remarks : BSDE

- $Z_n(t_{k-1})$ coincides with $\mathbb{E} \left[Y_n(t_k) \cdot \frac{W(t_k) - W(t_{k-1})}{t_k - t_{k-1}} \middle| \mathcal{F}_{t_{k-1}}^W \right]$

- $\underline{t} = t_{k-1}$ for $t_{k-1} \leq t < t_k$, (Y_n, Z_n) is solution of :

$$dY_n(t) = f(\underline{t}, {}_n X_{\underline{t}}, Y_n(\underline{t}), Z_n(\underline{t})) dt - Z_n(t) dW(t), Y_n(T) = g({}_n X_T)$$

- New numerical method ?

$Y_n \rightsquigarrow$ conditional expectation

$Z_n \rightsquigarrow$ vertical derivative of Y_n

Thank you for your attention !