A risk management approach to capital allocation

Khalil Said
PhD supervisors:
Mme Véronique Maume-Deschamps
M. Didier Rullière

Laboratoire de sciences actuarielle et financière (SAF) EA2429

Colloque Jeunes Probabilistes et Statisticiens
Les Houches, April 19, 2016
1. Introduction
2. Optimal allocation
3. Coherence properties
4. Discussion
5. Conclusion
Introduction

Multivariate risk theory:
- Dependence modeling;
- Multivariate ruin probabilities;
- Multivariate risk measures...

What is a capital allocation?
- Euler and Shapley principles ([Tasche, 2007],[Denault, 2001]).
- Minimization of some ruin probabilities or multivariate risk indicators.
What is a capital allocation?

**Figure**: What is a capital allocation?
Optimal allocation

- Multivariate risk indicators
- The allocation method
- Optimality conditions
- Penalty functions
We consider a vectorial risk process \( X^p = (X^p_1, \ldots, X^p_d) \), where \( X^p_k \) corresponds to the losses of the \( k^{th} \) business line during the \( p^{th} \) period.

We denote by \( R^p_k \) the reserve of the \( k^{th} \) line at time \( p \), so:

\[
R^p_k = u_k - \sum_{l=1}^{p} X^l_k,
\]

where \( u_k \in \mathbb{R}^+ \) is the initial capital of the \( k^{th} \) business line;

\( u = u_1 + \cdots + u_d \) is the initial capital of the group;

\( d \) is the number of business lines.

\( U^d_u = \{ v = (v_1, \ldots, v_d) \in [0, u]^d, \sum_{i=1}^{d} v_i = u \} \) is the set of possible allocations of the initial capital \( u \).

\( \forall i \in \{1, \ldots, d\} \) let \( \alpha_i = \frac{u_i}{u} \), then, \( \sum_{i=1}^{d} \alpha_i = 1 \) if \( (u_1, \ldots, u_d) \in U^d_u \).

\( X_k \) corresponds to the losses of the \( k^{th} \) branch during one period \((n = 1)\).
Cénac et al. (2012) defined the two following multivariate risk indicators, for \( d \) risks and \( n \) periods, given penalty functions \( g_k, \ k \in \{1, \ldots, d\} \):

- the indicator \( I \):

\[
I(u_1, \ldots, u_d) = \sum_{k=1}^{d} \mathbb{E} \left( \sum_{p=1}^{n} g_k(R_p^k) \mathbb{I}\{R_p^k < 0\} \mathbb{I}\{\sum_{j=1}^{d} R_j^p > 0\} \right),
\]

- the indicator \( J \):

\[
J(u_1, \ldots, u_d) = \sum_{k=1}^{d} \mathbb{E} \left( \sum_{p=1}^{n} g_k(R_p^k) \mathbb{I}\{R_p^k < 0\} \mathbb{I}\{\sum_{j=1}^{d} R_j^p < 0\} \right),
\]

\( g_k : \mathbb{R}^- \rightarrow \mathbb{R}^+ \) are \( C^1 \), convex functions with \( g_k(0) = 0, \ g_k(x) \geq 0, \ k = 1, \ldots, d \), \( g_k \) are decreasing functions on \( \mathbb{R}^- \).
Optimal allocation
Multivariate risk indicators

**Figure**: Multivariate risk indicators
The allocation method
Since the new regulation, such as Solvency 2, require a one year allocation strategy, in this paper we focus on a single period \((n = 1)\).

**Definition : Optimal allocation**

Let \(X\) be a non negative random vector of \(\mathbb{R}^d\), \(u \in \mathbb{R}^+\) and \(K_X : U_u^d \to \mathbb{R}^+\) a multivariate risk indicator associated to \(X\) and \(u\). An optimal allocation of the capital \(u\) for the risk vector \(X\) is defined by:

\[
(u_1, \ldots, u_d) \in \arg \inf \{K_X(v_1, \ldots, v_d)\}.
\]

- For risk indicators of the form \(K_X(v) = \mathbb{E}[S(X, v)]\), with a scoring function \(S : \mathbb{R}^{+d} \times \mathbb{R}^{+d} \to \mathbb{R}^+\), this definition can be seen as an extension in a multivariate framework of the concept of elicitation.
- For an initial capital \(u\), and an optimal allocation minimizing the multivariate risk indicator \(I\), we seek \(u^* \in \mathbb{R}^d\) such that:

\[
I(u^*) = \inf_{v_1 + \cdots + v_d = u} I(v), \quad v \in \mathbb{R}^d_+.
\]
Assumptions

H1 $\mathcal{K}_X$ admits a unique minimum in $\mathcal{U}_u^d$. In this case, we denote by $A_{X_1,\ldots,X_d}(u) = (u_1, \ldots, u_d)$ the optimal allocation of $u$ on the $d$ risky branches in $\mathcal{U}_u^d$.

H2 The functions $g_k$ are differentiable and such that for all $k \in \{1, \ldots, d\}$, $g'_k(u_k - X_k)$ admits a moment of order one, and $(X_k, S)$ has a joint density distribution denoted by $f(X_k, S)$.

H3 The $d$ risks have the same penalty function $g_k = g, \forall k \in \{1, \ldots, d\}$.

The first assumption is verified when the indicator is strictly convex, this is particularly true if at least one function $g_k$ is strictly convex; and the joint density $f(X_k, S)$ support contains $[0, u]^2$. 
Under assumption H2, the risk indicators $I$ and $J$ are differentiable,

$$(
\nabla I(v))_i = \sum_{k=1}^{d} \int_{v_k}^{+\infty} g_k(v_k - x)f_{X_k,S}(x, u)dx + \mathbb{E}\left[g_i'(v_i - X_i) \mathbb{1}_{\{X_i > v_i\}} \mathbb{1}_{\{S \leq u\}}\right]$$

and,

$$(
\nabla J(v))_i = \sum_{k=1}^{d} \int_{v_k}^{+\infty} g_k(v_k - x)f_{X_k,S}(x, u)dx + \mathbb{E}\left[g_i'(v_i - X_i) \mathbb{1}_{\{X_i > v_i\}} \mathbb{1}_{\{S \geq u\}}\right].$$

Under H1 and H2, using Lagrange multipliers, we obtain an optimality condition verified by the unique solution,

$$\mathbb{E}\left[g_i'(u_i - X_i) \mathbb{1}_{\{X_i > u_i\}} \mathbb{1}_{\{S \leq u\}}\right] = \mathbb{E}\left[g_j'(u_j - X_j) \mathbb{1}_{\{X_j > u_j\}} \mathbb{1}_{\{S \leq u\}}\right], \quad \forall (i,j) \in \{1, \ldots, d\}^2$$
Penalty functions
Ruin severity as penalty function

- A natural choice for penalty functions is the ruin severity: \( g_k(x) = |x| \).
- If the joint density \( f(x_k, s) \) support contains \([0, u]^2\), for at least one \( k \in \{1, \ldots, d\} \), our optimization problem has a unique solution.
- We may write the indicators as follows:

\[
I(u_1, \ldots, u_d) = \sum_{k=1}^{d} \mathbb{E} \left( (X_k - u_k) + \mathbbm{1}_{\{S \leq u\}} \right),
\]

and,

\[
J(u_1, \ldots, u_d) = \sum_{k=1}^{d} \mathbb{E} \left( (X_k - u_k) + \mathbbm{1}_{\{S \geq u\}} \right).
\]

- The optimality condition:

\[
P(X_i > u_i, S \leq u) = P(X_j > u_j, S \leq u), \quad \forall (i, j) \in \{1, 2, \ldots, d\}^2.
\]

For the \( J \) indicator, this condition can be written:

\[
P(X_i > u_i, S \geq u) = P(X_j > u_j, S \geq u), \quad \forall (i, j) \in \{1, 2, \ldots, d\}^2.
\]
Coherence properties

- Coherence
- Other desirable properties
- Coherence of the optimal allocation
Coherence


**Definition : Coherent allocation**

A capital allocation \((u_1, \ldots, u_d) = A_{X_1, \ldots, X_d}(u)\) of an initial capital \(u \in \mathbb{R}^+\) is coherent if it satisfies the following properties :

1. **Full allocation** :
   \[
   \sum_{i=1}^{d} u_i = u.
   \]

2. **Riskless allocation** : For a deterministic risk \(X = c\), where the constant \(c \in \mathbb{R}^+\) :
   \[
   A_{X, X_1, \ldots, X_d}(u) = (c, A_{X_1, \ldots, X_d}(u - c)).
   \]
Definition: Coherent allocation

3. Symmetry: if

\[(X_1, \ldots, X_{i-1}, X_i, X_{i+1}, \ldots, X_{j-1}, X_j, X_{j+1}, \ldots, X_d)\]
\[\mathcal{L} = (X_1, \ldots, X_{i-1}, X_j, X_{i+1}, \ldots, X_{j-1}, X_i, X_{j+1}, \ldots, X_d),\]

then \(u_i = u_j\).

4. Sub-additivity: \(\forall M \subseteq \{1, \ldots, d\}\), let

\[(u^*, u_1^*, \ldots, u_r^*) = A\sum_{i \in M} X_i, x_j \in \{1, \ldots, d\} \setminus M (u), \]

where \(r = d - \text{card}(M)\)

and \((u_1, \ldots, u_d) = A_{X_1, \ldots, X_d} (u) : \]

\[u^* \leq \sum_{i \in M} u_i.\]

5. Comonotonic additivity: For \(r \leq d\) comonotonic risks,

\[A_{X_i \in \{1, \ldots, d\} \setminus CR, \sum_{k \in CR} X_k (u)} = (u_{i \in \{1, \ldots, d\} \setminus CR}, \sum_{k \in CR} u_k),\]

where \(CR\) denotes the set of the \(r\) comonotonic risk indexes.
Desirable properties

**Positive homogeneity**

An allocation is positively homogeneous, if for any $\alpha \in \mathbb{R}^+$, it satisfies:

$$A_{\alpha X_1, \ldots, \alpha X_d}(\alpha u) = \alpha A_{X_1, \ldots, X_d}(u).$$

**Translation invariance**

An allocation is invariant by translation, if for all $(a_1, \ldots, a_d) \in \mathbb{R}^d$ such that $u > \sum_{k=1}^{d} a_k$, it satisfies:

$$A_{X_1 + a_1, \ldots, X_d + a_d}(u) = A_{X_1, \ldots, X_d}(u - \sum_{k=1}^{d} a_k) + (a_1, \ldots, a_d).$$

**Continuity**

An allocation is continuous, if for all $i \in \{1, \ldots, d\}$:

$$\lim_{\epsilon \to 0} A_{X_1, \ldots, (1+\epsilon)X_i, \ldots, X_d}(u) = A_{X_1, \ldots, X_i, \ldots, X_d}(u).$$
Desirable properties

- We recall the definition of the order stochastic dominance, as it is presented in Shaked and Shanthikumar (2007) [Shaked and Shanthikumar, 2007].
- For random variables $X$ and $Y$, $Y$ first-order stochastically dominates $X$ if and only if:
  \[ \bar{F}_X(x) \leq \bar{F}_Y(x), \quad \forall x \in \mathbb{R}^+, \]
  and in this case we denote: $X \leq_{st} Y$.
- This definition is also equivalent to the following one:
  \[ X \leq_{st} Y \iff \mathbb{E}[u(X)] \leq \mathbb{E}[u(Y)], \text{for all } u \text{ increasing function.} \]

Monotonicity

An allocation satisfies the monotonicity property, if for $(i, j) \in \{1, \ldots, d\}^2$:
\[ X_i \leq_{st} X_j \Rightarrow u_i \leq u_j. \]
Coherence of the optimal allocation

In the case of penalty functions $g_k(x) = |x| \ \forall k \in \{1, \ldots, d\}$, and for continuous random vector $(X_1, \ldots, X_d)$, such that the joint density $f(x_k, s)$ support contains $[0, u]^2$, for at least one $k \in \{1, \ldots, d\}$, the optimal allocation by minimization of the indicators $I$ and $J$ is a symmetric riskless full allocation. It satisfies the properties of comonotonic additivity, positive homogeneity, translation invariance, monotonicity, and continuity.

General results are presented in Maume-Deschamps, Rullière and S (2016) [Maume-Deschamps et al., 2016b].

The optimal allocation method may be used for the economic capital allocation between the different branches of a group.
What could be the best choice for a capital allocation?

- The optimal allocation can be considered **coherent** from an economic point of view;
- The first goal of the Solvency 2 norms is the insurers’ protection;
- The classical methods of risk allocation give **weight** to each business line in the group risk.
- The optimal allocation is based on a **global risk optimization**.
- The capital allocation by minimizing a risk indicator seems more in line the with **Solvency 2 goals**.
- Conventional capital allocation methods are based on a chosen **univariate risk measure** and their properties are derived from those of this risk measure.
- It seems more coherent in a **multivariate framework** to use directly a multivariate risk indicator, not only for risk measurement, but also for capital allocation.
- Another important criterion in the choice of the allocation method is the **nature of the capital**.
- The best allocation method choice depends finally on the **risk aversion** of the insurer.
In this article, we have shown that the capital allocation method by minimization of some multivariate risk indicators can be considered as coherent from an economic point of view.

In the case of the proposed optimal capital allocation, the risk management is at the heart of the allocation process. That is why we think that from a risk management point of view, this method can be considered as more flexible.

Allocation by minimizing the indicators $I$ and $J$ is studied in higher dimension in Maume-Deshamps et al. (2016);

Its behavior and asymptotic behavior for some special distributions’ families are also analyzed in [Maume-Deschamps et al., 2016a];

The impact of dependence on the allocation composition is studied in the same paper.

Finally, the choice of a capital allocation method remains a complex and crucial exercise.
Thank you for your attention!

Khalil Said
PhD supervisors:
Mme Véronique Maume-Deschamps
M. Didier Rullière

Laboratoire de sciences actuarielle et financière (SAF) EA2429

Colloque Jeunes Probabilistes et Statisticiens
Les Houches, April 19, 2016


