Superconcentration inequalities for centered Gaussian stationnary processes

Kevin Tanguy

Université de Toulouse

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Outline

- What is superconcentration ?
- Convergence of extremes (Gaussian case).
- Superconcentration inequality for stationary Gaussian sequences.
- Tools and sketch of the proof.

- What is superconcentration ?

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In fact,

$$\mathrm{Var}(M_n) \leq \frac{C}{\log n}, \quad C > 0$$

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Chatterjee's terminology : superconcentration phenomenon.

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Lot of different models

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- Put X_e $i.i.d.$ $\mathcal{N}(0,1)$ on each edge e.
- Take a path π from the top to the bottom of the tree, $X_\pi = \sum_{e\in\pi} X_e$.

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- In fact, $\mathrm{Var}(\max_\pi X_\pi) \leq C$ [Bramson-Ding-Zeitouni].

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- Stationary Gaussian sequences.

- What is superconcentration ?
- Convergence of extremes (Gaussian case).

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Proposition [Chatterjee '14]

$$\mathrm{Var}(M_n) \leq \frac{C}{\log n}.$$

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- Superconcentration inequality for stationary Gaussian sequences.

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- Reflects asymptotics Gumbel

Recall

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- Reflects Gumbel asymptotics.
- Implies $\mathrm{Var}(\max_i X_i) \leq \frac{C}{\log n}$ (optimal).

Tools

Proof ?

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Chatterjee's scheme of proof for the variance at the exponential level.

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General theorem implies superconcentration inequality for Gaussian stationary sequences.

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- Semigroup representation of the variance
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- Proper use of a covering $\mathcal{C}$ of $\{1, \dots n\}$.

Conclusion

Similar results

- Gaussian stationary sequences with other covariance behavior.
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- Gaussian stationary processes on $\mathbb{R}^d$.
- discrete Gaussian free field on $\mathbb{Z}^d$, $d \geq 3$
  (infinite group of finite type+volume growth condition ok).
- Uniform measure on the sphere $\mathcal{S}^{n-1} \subset \mathbb{R}^n$.
- Log-concave measures on $\mathbb{R}^n$ with convexity assumptions.

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Hypercontractivity relevant Gaussian processes $\simeq$ independent case (variance $\sim \frac{1}{\log n}$).

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discrete Gaussian free field on $\mathbb{Z}^2$ completely different behavior.

- $\mathrm{Var}(M_n) = O(1)$ [Bramson-Ding-Zeitouni]
- convergence in distribution Gumbel randomly shifted [Bramson-Ding-Zeitouni '15].

Hypercontractivity alone doesn't work .

Thank you for your attention.