Almost-sure hedging under permanent price impact

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April 20, 2016

Outline

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2. Process dynamics and hedging problems
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   - Hedging problem: covered options
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   - Statement of pricing PDE
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- Approach:
  1. Define a continuous time trading dynamics from a discrete time rule.
  2. Provide the PDE characterization: stochastic target tool.

Impact rule & Trading signal

Basic rule: an order δ moves the price by

$$X_{t-} \longrightarrow X_t = X_{t-} + \delta f(X_{t-}),$$

and costs

$$\delta X_{t-} + \frac{1}{2}\delta^2 f(X_{t-}) = \delta(\frac{X_{t-} + X_t}{2}).$$

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A trading signal is an Itô process of the form

$$Y = Y_0 + \int_0^{\cdot} b_s ds + \int_0^{\cdot} a_s dW_s.$$

Discrete trading dynamics

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- Dynamics of the wealth and of the asset: discrete trading and pass to the limit.

Continuous trading dynamics

Passing to the limit $n \to \infty$, we have the following convergence in $\mathbf{S}_2$

$$Y = Y_0 + \int_0^\cdot b_s ds + \int_0^\cdot a_s dW_s$$

$$X = X_0 + \int_0^\cdot \sigma(X_s) dW_s + \int_0^\cdot f(X_s) dY_s + \int_0^\cdot (\mu + a_s \sigma f')(X_s) ds$$

$$V = V_0 + \int_0^\cdot Y_s dX_s + \frac{1}{2} \int_0^\cdot a_s^2 f(X_s) ds,$$

at a speed $\sqrt{n}$, where

$$V = \text{ cash part } + YX = \text{“portfolio value”}.$$

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- Gamma constraint:
  Re-write the dynamics of Y as below:

$$dY = a\,dW + b\,dt = \gamma^a(X)\,dX + \mu_Y^{a,b}(X)\,dt$$

with $\gamma^a = a/(\sigma + af)$. Then $\gamma^a(X)$ is bounded by some function.

Mathematical definition

Super-hedging price under gamma constraint:

$$\mathrm{v}(t,x) := \inf\{v = c + yx : (c,y) \in \mathbb{R}^2 \text{ s.t. } \mathcal{G}^\gamma(t,x,\mathrm{v},y) \neq \emptyset\}$$

where $\mathcal{G}^\gamma(t,x,\mathrm{v},y)$ is the set of (a,b) s.t. $\phi := (y,a,b)$ satisfies:

$$V_T^{t,x,\mathrm{v},\phi} \geq g(X_T^{t,x,\phi}), \quad \text{and} \quad \gamma^a(X) \leq \bar{\gamma}(X)$$

with $f\bar{\gamma} < 1 - \epsilon$, $\epsilon > 0$.

Informal derivation

If we follow the delta-hedging rule:

$$V = \mathrm{v}(\cdot, X)\ , \text{ and } Y = \partial_x \mathrm{v}(\cdot, X)$$

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Equating the dt terms on $V = \mathrm{v}(\cdot, X)$ & dX terms on $Y = \partial_x \mathrm{v}(\cdot, X)$:

$$\begin{aligned} \frac{1}{2}a^2 f(X) &= \partial_t \mathrm{v}(\cdot, X) + \frac{1}{2}(\sigma + af)^2(X)\partial^2_{xx}\mathrm{v}(\cdot, X) \\ \gamma^a &= \partial^2_{xx}\mathrm{v}(\cdot, X) \end{aligned}$$

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By definition of γ^a and some calculate:

$$\left[-\partial_t \mathrm{v} - \frac{1}{2} \frac{\sigma^2}{(1 - f\partial_{xx}^2 \mathrm{v})} \partial_{xx}^2 \mathrm{v} \right](\cdot, X) = 0$$

Gamma constraint insures the non-singularity of the PDE.

Pricing PDE and Terminal condition

- Pricing PDE: on $[0,T) \times \mathbb{R}$

$$
\begin{aligned}
F[\mathrm{v}_{\bar\gamma}](t,x) &:= \min\left\{ -\partial_t \mathrm{v}_{\bar\gamma} - \frac{1}{2}\frac{\sigma^2}{(1 - f\partial^2_{xx}\mathrm{v}_{\bar\gamma})}\partial^2_{xx}\mathrm{v}_{\bar\gamma}, \bar\gamma - \partial^2_{xx}\mathrm{v}_{\bar\gamma} \right\} \\
&= 0
\end{aligned}
$$

- Propagation of the gamma constraint to the boundary:

$$\mathrm{v}_{\bar\gamma}(T-,\cdot) = \hat g \text{ on } \mathbb{R}$$

where $\hat g$ is the smallest (viscosity) super-solution of

$$\min\{\varphi - g, \bar\gamma - \partial^2_{xx}\varphi\} = 0$$

Super-solution property

Theorem (Geometric Dynamic Programming Principle)

If $V_0 > \mathrm{v}(0, X_0)$, *then there exists* (a, b, Y_0) *such that*

$$V_\theta \geq \mathrm{v}(\theta, X_\theta)$$

for any stopping time $\theta \in [0, T]$.

Result: A super-solution $\underline{\mathrm{v}}_{\bar{\gamma}} \leq \mathrm{v}_{\bar{\gamma}}$. cf. N.Touzi, M.Soner.

Sub-solution property: shaken operator

Objective: Construct a solution $\bar{\mathrm{v}}_{\bar{\gamma}} \geq \mathrm{v}_{\bar{\gamma}}$.

Theorem

Define the shaken operator

$$F^{\epsilon}[\varphi](t,x) := \min_{x' \in B_{\epsilon}(x)} \min \left\{ -\partial_t \varphi + \frac{\sigma^2(x')}{2(1 - f(x')\partial^2_{xx}\varphi)}, \bar{\gamma}(x') - \partial^2_{xx}\varphi \right\}(t,x)$$

then $\forall \epsilon > 0$, $\exists \bar{\mathrm{v}}^{\epsilon}_{\bar{\gamma}}$ *the unique continuous viscosity solution of*

$$F^{\epsilon}[\varphi](t,x)\mathbb{1}_{[0,T)} + [\varphi - (\hat{g} + \epsilon)]\mathbb{1}_{\{T\}} = 0$$

Moreover, $\bar{\mathrm{v}}^{\epsilon}_{\bar{\gamma}} \geq \hat{g} + \epsilon/2$ *on* $[T - c_{\epsilon}, T] \times \mathbb{R}$.

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By the stability of viscosity solution:

$$\bar{\mathrm{v}}^{\epsilon}_{\bar{\gamma}} \to \bar{\mathrm{v}}_{\bar{\gamma}}, \quad \text{as } \epsilon \to 0$$

where $\bar{\mathrm{v}}_{\bar{\gamma}}$ is the unique viscosity solution of

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Remain to prove: $\bar{\mathrm{v}}_{\bar{\gamma}} \geq \mathrm{v}_{\bar{\gamma}}$.

Sub-solution property: regularization

Theorem

For $\psi \in C^\infty$, define $\psi_\delta = \delta^{-2}\psi(\cdot/\delta)$. Then $\bar{\mathrm{v}}^{\epsilon,\delta}_{\bar{\gamma}} := \bar{\mathrm{v}}^{\epsilon}_{\bar{\gamma}} \star \psi_\delta$ is a super-solution of

$$F[\bar{\mathrm{v}}^{\epsilon,\delta}_{\bar{\gamma}}](t,x)\mathbb{1}_{[\delta,T)} + (\bar{\mathrm{v}}^{\epsilon,\delta}_{\bar{\gamma}} - \hat{g})\mathbb{1}_{\{T\}} = 0$$

Then $Y := \partial_x \bar{\mathrm{v}}^{\epsilon,\delta}_{\bar{\gamma}}(\cdot, X)$ provides a super-hedging strategy, i.e. $\bar{\mathrm{v}}^{\epsilon,\delta}_{\bar{\gamma}} \geq \mathrm{v}_{\bar{\gamma}}$. Moreover, $\lim\limits_{\epsilon,\delta\to 0} \bar{\mathrm{v}}^{\epsilon,\delta}_{\bar{\gamma}} = \bar{\mathrm{v}}_{\bar{\gamma}}$.

Conclusion:

$$\bar{\mathrm{v}}_{\bar{\gamma}} = \lim_{\epsilon,\delta\to 0} \bar{\mathrm{v}}^{\epsilon,\delta}_{\bar{\gamma}} \geq \mathrm{v}_{\bar{\gamma}} \geq \underline{\mathrm{v}}_{\bar{\gamma}} \geq \bar{\mathrm{v}}_{\bar{\gamma}}$$

Numerical res: Bachelier model with constant impact

Model: $dX_t = 0.2\, dW_t$; Butterfly: $g(x) = (x+1)^+ - 2x^+ + (x-1)^+$

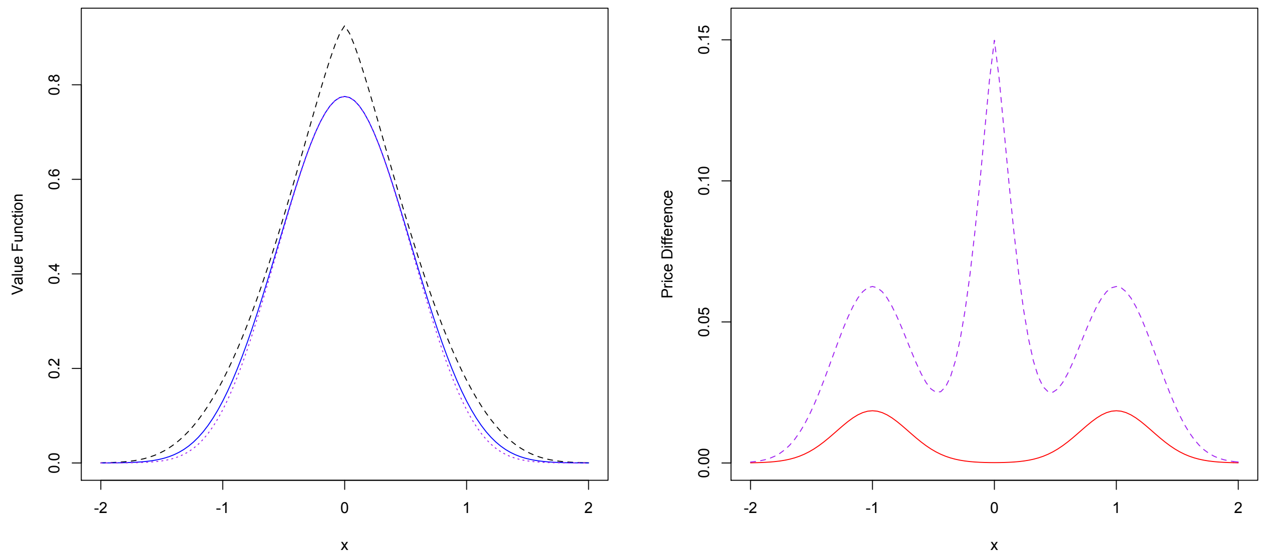

Figure 1: Left: Super-hedging price. Dashed line: $\lambda = 0.5$, $\bar{\gamma} = 1.75$; solid line: $\lambda = 0$, $\bar{\gamma} = 1.75$; dotted line: $\lambda = 0$, $\bar{\gamma} = +\infty$. Right: Difference with the price associated to $\lambda = 0$, $\bar{\gamma} = +\infty$. Dashed line: $\lambda = 0.5$, $\bar{\gamma} = 1.75$; solid line: $\lambda = 0$, $\bar{\gamma} = 1.75$.

Thank you very much!