Gaussian measures and fractional Brownian fields

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Plan

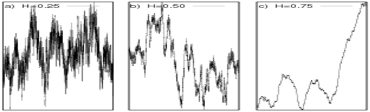
1. Introduction: fractional Brownian fields
2. Gaussian measures and the fractional Brownian field
   - Gaussian measures on a Banach space
   - Wiener space of the fractional Brownian motion
   - Regularity of the fractional Brownian field
3. The multiparameter fractional Brownian motion
   - Small balls of the fBf
   - Geometric properties of the mpfBm

One-parameter case

Hurst parameter, $h \in (0, 1)$.

1. fractional Brownian motion

$$R_h(t,s) = \frac{1}{2}(t^{2h} + s^{2h} - |t-s|^{2h})$$



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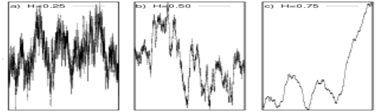

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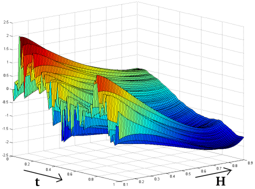

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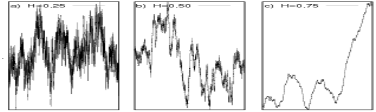

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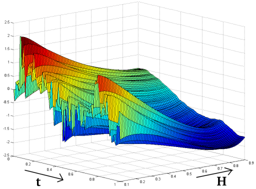
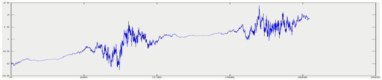


3. multifractional Brownian motion

$$t \mapsto h(t)$$



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- Fractional Brownian sheet: for $h = (h_1, \dots, h_n)$,

$$W_t^h = \int_{\mathbb{R}^N} \prod_{k=1}^{N} \left(|t_k - s_k|^{h_k - 1/2} - |s_k|^{h_k - 1/2} \right) \mathrm{d}\mathbb{W}_s$$

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- The multiparameter fBm [Herbin & Merzbach 06].

Multiparameter fBm

$(T, \mathscr{T}, m)$ is a measurable space, $f, g \in L^2(T, m)$, $h \in (0, 1/2]$:

$$k_h : (f, g) \mapsto \frac{1}{2}\left(m(f^2)^{2h} + m(g^2)^{2h} - m((f-g)^2)^{2h}\right)$$

is positive definite (where $m(f^2) = \|f\|^2_{L^2(m)}$).

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Definition

The multiparameter fBm is a centred Gaussian process with covariance:

$$k_h(t, s) = \frac{1}{2}\left(\lambda([0, t])^{2h} + \lambda([0, s])^{2h} - \lambda([0, t] \triangle [0, s])^{2h}\right)$$

Applications of multiparameter Brownian fields

- Lévy fBm: its sample paths properties, e.g. [Pitt 78], [Talagrand 95], [Xiao 97], ...
- fractional Brownian sheet: [Kamont 96],
  - *sample paths properties*, e.g. [Ayache & Xiao 05]
  - *stochastic calculus*, after the seminal work of Cairoli and Walsh on the Brownian sheet[Cairoli & Walsh 75], extensions to stochastic integrals in the plane wrt general Gaussian processes [Dalang 99] [Balan & Conus 13], pathwise approach [Mishura & Ilchenko 06], and Malliavin approach [Tudor & Viens 03], ...
- mpfBm, [Herbin & Merzbach 06], [Herbin & Xiao 14] + articles in preparation.

Plan

Gaussian measures

Definition

A *Gaussian measure* μ on a separable Banach space E is a measure under which any continuous linear functional $x^* \in E^*$ has a Gaussian law.

Let H_μ be the Cameron-Martin Space of μ, defined as

$$H_\mu = \overline{E^*}^{L^2(\mu)},$$

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or equivalently, using the canonical embedding S given by:

$$Sx^* = \int_E x\langle x^*, x\rangle \,\mathrm{d}\mu(x) .$$

H_μ is densely and compactly embedded into E, and (H_μ, E, μ) is called a *Wiener space*.

Proposition (see e.g. [Stroock 10])

Let H and H_μ be two separable Hilbert spaces, H_μ being endowed with a Wiener space structure (H_μ, E, μ). H can also be endowed with such a structure by isometry, i.e. if $u : H_\mu \to H$ is a linear isometry, $(H, \tilde{u}(E), \tilde{u}_* \mu)$ is a Wiener space.

Wiener space of the fBm

- Standard Wiener space (h=1/2): $\mathscr{W} = \mathscr{L}(B)$ where B is a standard Brownian motion considered as a random variable taking values in $C_0[0,1]$.

$$H_{\mathscr{W}} = H_0^1 = \left\{ f(t) = \int_0^t \dot{f}(s)\mathrm{d}s,\ \dot{f} \in L^2[0,1] \right\}.$$

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- For any $h \in (0,1)$, the Wiener space of the fBm is given by:

$$(\mathscr{I}_{0+}^{h+1/2}(L^2[0,1]), C_0[0,1], \mathscr{W}_h).$$

Representation of the fractional Brownian field

On $[0,1]$, the fBm can be represented as [Decreusefond et al. 99]:

$$B_{h,t} = \int_{[0,1]} K_h(t,s)\ \mathrm{d}B_s\ ,$$

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We proved that there is an operator $\mathscr{K}_h : H_h \to (C_0[0,1])^*$ such that:

$$B_{h,t} = \int_{C_0[0,1]} \langle \mathscr{K}_h k_h(\mathbf{1}_{[0,t]},\cdot), x\rangle\ \mathrm{d}\mathbb{B}_x\ ,$$

where $\mathbb{B}$ is a white noise on $C_0[0,1]$ of control measure $\mathscr{W}$.

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Proposition

For any $h \in (0,1/2]$, there exists $(H(k_h), E_h, \mu_h)$ a Wiener space and an operator $\tilde{\mathscr{K}}_h$ of $H(k_h) \to E^*$ from which can be defined:

$$B_{h,f} = \int_E \langle \tilde{\mathscr{K}}_h k_h(f,\cdot), x\rangle \, \mathrm{d}\mathbb{B}_x \ ,$$

and for any fixed h, $\{B_{h,f}, f \in L^2\}$ is a h-fBm with covariance k_h.

Regularity of the increments

Theorem

There exists a fractional Brownian field indexed over $(0,1/2]\times L^2(T,m)$ whose covariance satisfies: for any $\eta\in(0,1/4)$ and any compact subset D of $L^2(T,m)$, there exists $C_{\eta,D}\equiv C>0$ such that for any $f,f'\in D$, and any $h,h'\in[\eta,1/2-\eta]$,

$$\mathbb{E}\left(B_{h,f}-B_{h',f'}\right)^2\leq C_1(h-h')^2+C_2\ m\left((f-f')^2\right)^{2(h\wedge h')}\ .$$

Application to the continuity of the sample paths

Definition (Totally bounded)

A metric space $(\mathscr{S}, d)$ is *totally bounded* if for every $\varepsilon > 0$, $\mathscr{S}$ can be covered by a finite number of balls of radius less than ε. The *metric entropy* $N(\varepsilon), \varepsilon > 0$ is the smallest number of such balls.

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Theorem ([Dudley 73])

Let X be a centered Gaussian field indexed by $\mathscr{S}$ and define the pseudo-distance d_X by: $d_X(s,t)^2 = \mathbb{E}(X_s - X_t)^2 \quad \forall s,t \in \mathscr{S}$.

Assume that $\mathscr{S}$ is d_X-compact and that:

$$\int_0^1 \sqrt{\log(N(\mathscr{S}, d_X, \varepsilon))}\, d\varepsilon < \infty \ .$$

Then X has a continuous version on $\mathscr{S}$.

Continuity of the fBf

Let d_m be the distance on $L^2(T, m)$.

Proposition

Let K be a compact subset with nonempty interior of $L^2(T, m)$. If

$$\int_0^1 \sqrt{\log N(K, d_m, \varepsilon)}\, d\varepsilon < \infty\ ,$$

then $\{B_{h,f}, (h,f) \in (0,1/2] \times K\}$ has a continuous modification.

Hölder continuity of the sample paths

Let $\boldsymbol{h} : t \in [0,1]^N \mapsto h(t) \in (0, 1/2]$ and define the *multiparameter multifractional Brownian motion* from the fBf B on $L^2([0,1]^N, Leb.)$ as:

$$\forall\, t \in [0,1]^N, \quad B_t^{\boldsymbol{h}} = B_{h(t), \mathbf{1}_{[0,t]}} .$$

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The local Hölder regularity of a function f can be measured by:

- the pointwise coefficient:

$$\alpha_f(t_0) = \sup\left\{\alpha : \limsup_{r\to 0} \sup_{s,t\in B(t_0,r)} \frac{|f(t)-f(s)|}{r^\alpha} < \infty\right\} ,$$

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Hölder continuity of the sample paths

Theorem

Let us assume that for any $t \in [0,1]^N$, $\boldsymbol{h}(t)$ *is bigger than its regularity, i.e.* $\boldsymbol{h}(t) \geq \alpha_{\boldsymbol{h}}(t)$. *Then, almost surely,*

$$\forall\, t \in [0,1]^N, \quad \alpha_{B^{\boldsymbol{h}}}(t) = \tilde{\alpha}_{B^{\boldsymbol{h}}}(t) = \boldsymbol{h}(t)\ .$$

Plan

Small balls of the fBf

Theorem

Let $h \in (0, 1/2)$, K *a compact set in* $L^2(T, m)$. *Then, for some* $k_1 > 0$,

$$\mathbb{P}\left(\sup_{f \in K} |B_f^h| \leq \varepsilon\right) \leq \exp\left(-k_1 \ N(K, d_h, \varepsilon)\right) \ ,$$

and if there exists ψ *such that for any* $\varepsilon > 0$, $N(K, d_h, \varepsilon) \leq \psi(\varepsilon)$ *and* $\psi(\varepsilon) \approx \psi(\varepsilon/2)$, *then for some constant* $k_2 > 0$,

$$\mathbb{P}\left(\sup_{f \in K} |B_f^h| \leq \varepsilon\right) \geq \exp\left(-k_2 \ \psi(\varepsilon)\right) \ .$$

General remarks on the mpfBm

B_t^h will denote a multiparameter fBm. Recall that

$$k_h(t,s) = \frac{1}{2}\left(\lambda([0,t])^{2h} + \lambda([0,s])^{2h} - \lambda([0,t] \triangle [0,s])^{2h}\right).$$

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- B^h is not increment stationary;
- for $a > 0$, d_{B^h} is equivalent to the Euclidean distance on $[a,1]^N$;
- away from 0, modulus of continuity and Hausdorff measure of the graph are similar to Lévy fBm.

Small balls of the multiparameter fBm

Lemma

For $h < 1/2$, there are constants $k_1 > 0$ and $k_2 > 0$ such that for any fixed $r \in (0,1)$ and ε small enough (compared to r),

$$\exp\left\{-k_2 \frac{r^{2N}}{\varepsilon^{N/h}}\right\} \leq \mathbb{P}\left(\sup_{t\in[0,r]^N} |B_t^h| \leq \varepsilon\right) \leq \exp\left\{-k_1 \frac{r^{2N}}{\varepsilon^{N/h}}\right\}$$

For every $h \in (0,1/2)$, let us denote $M^h(r) = \sup_{t\in[0,r]^N} |B_t^h|$, $r \in [0,1]$. We will also need $\psi_h(r) = r^{2h}\left(\log\log(r^{-1})\right)^{-h/N}$.

Theorem

Let $h \in (0,1/2)$ and let M^h and ψ_h be as defined above. Then there exists a constant $c \in (0,\infty)$ such that, almost surely:

$$\liminf_{r\to 0} \frac{M^h(r)}{\psi_h(r)} = \beta \ .$$

Ideas of the proof

- Find $(r_k^{(1)})_{k\in\mathbb{N}}$ and $\beta^{(1)}$ s.t.:

$$\sum_k \mathbb{P}\left(M_h(r_k^{(1)})/\psi_h(r_k^{(1)}) \le \beta^{(1)}\right) < \infty$$

  using the small ball probabilities;

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- Spectral representation: for $h < 1/2$, there exists $\Delta^h(dx)$ on E such that:

$$\|Sx^*\|_H^{2h} = \int_E \left(1 - \cos\langle x^*, x\rangle\right)\ \Delta^h(dx)\ .$$

$$\Rightarrow B_t^h = \frac{1}{\sqrt{2}} \int_E \left(1 - e^{i\langle k_{1/2}(\mathbf{1}_{[0,t]},\cdot),x\rangle}\right)\ \mathrm{d}\mathbb{B}_x^{\Delta}$$

Thank you for your attention.

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Reproducing Kernel Hilbert Space

Definition

Let (T, m) be a complete separable metric space and R a continuous covariance function on $T \times T$. There exists a unique Hilbert space $H(R)$ such that:

1. $H(R)$ is a space of functions from $T \to \mathbb{R}$, and for all $t \in T$, $R(., t) \in H(R)$;
2. the scalar product is given by: $\forall t \in T, \forall f \in H(R)$,

$$\big(f, R(., t)\big)_{H(R)} = f(t).$$

This is a *separable* Hilbert space. It satisfies $H(R) = \overline{\mathrm{Span}\{R(., t), t \in T\}}^{\|.\|_{H(R)}}$.

Local nondeterminism

For any fixed $h < 1/2$, the fractional Brownian field is locally nondeterministic:

Lemma

Let $h \in (0, 1/2)$. There exists a positive constant C_0 such that for all $f \in L^2(T, m)$ and for all $r \le \|f\|$, the following holds:

$$\mathrm{Var}\left(B_f^h \mid B_g^h, \|f - g\| \ge r\right) = C_0 r^{2h}.$$