Sharp minimax test for large covariance matrices

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Plan

- Let $X_1, \ldots, X_n$, be n independent and identically distributed p-vectors, $X_k = (X_{k,1}, \ldots, X_{k,p})^\top$ for all $k = 1, \ldots, n$, following a multivariate normal distribution $\mathcal{N}_p(0, \Sigma)$.

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- We assume that $n, p \longrightarrow +\infty$.
- Goodness-of-fit test problem:

$$\begin{aligned} H_0 &: \Sigma = Id \\ H_1 &: \Sigma \neq Id \quad \text{where } Id \text{ is the } p \times p \text{ identity matrix} \end{aligned}$$

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- The distance between H_0 and H_1 can be evaluated by considering different norms. For example : $\|\Sigma - Id\|_F \geq \varphi, \|\Sigma - Id\|_q \geq \varphi$, for $q \in \mathbb{N}$.

- We restrict the set of matrices under the alternative to the collection of matrices whose elements decrease polynomially when moving away from the diagonal:

$$\mathcal{F}(\alpha, L) = \{\Sigma \in C_{>0}\,; \frac{1}{p}\sum_{i=1}^{p}\sum_{\substack{j=1\\ j>i}}^{p} \sigma_{ij}^2 |i-j|^{2\alpha} \leq L, \, \forall p \text{ and } \sigma_{ii} = 1\}$$

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- Testing problem :

$$\begin{aligned} H_0 &: \quad \Sigma = I \\ H_1 &: \quad \Sigma \in \mathcal{F}(\alpha, L), \quad \text{such that } \frac{1}{2p}\|\Sigma - I\|_F^2 \geq \varphi^2. \end{aligned}$$

Denote $Q(\alpha, L, \varphi) = \{\Sigma \in \mathcal{F}(\alpha, L)\,;\, \frac{1}{p}\sum_{i=1}^{p}\sum_{\substack{j=1\\ j>i}}^{p} \sigma_{ij}^2 \geq \varphi^2\ \}$

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- Class of Toeplitz matrices :

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- Remark : heuristically σ_k^2 replaces $\dfrac{1}{p-k}\displaystyle\sum_{j=1}^{p-k} \sigma_{j(j+k)}^2$

- Recall that a stationary Gaussian process $X_j,\ j \geq 1$ with covariances $\sigma_k = cov(X_j, X_{j+k})$, has spectral density f, given by :

$$f(x) = \frac{1}{2\pi}\left(\sigma_0 + 2\sum_{k=1}^{\infty}\sigma_k cos(kx)\right) \text{ for } x \in [-\pi, \pi]$$

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- Ermakov (1994), gives the sharp minimax rate for the following testing problem associated to the spectral density :

$$H_0 : f = f_0 \quad \text{v.s} \quad H_1 : f \neq f_0 \text{ such that } \|f - f_0\|_2 \geq \varphi.$$

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- Gobulev, Nussbaum and Zhou (2010 *Ann. Stat.*) gives the adaptive testing rate for the spectral density model using the asymptotic equivalence to Gaussian white noise.

- A test ψ is a measurable function with respect to the observations, taking values in $\{0,1\}$. The total error probability of ψ

$$\gamma(\psi, Q(\alpha, L, \varphi)) = \eta(\psi) + \beta(\psi, Q(\alpha, L, \varphi))$$

where, type I error probability

$$\eta(\psi) = \mathbb{P}_I(\psi = 1)$$

and maximal type II error probability

$$\beta(\psi, Q(\alpha, L, \varphi)) = \sup_{\{\Sigma \in Q(\alpha, L, \varphi)\}} \mathbb{P}_\Sigma(\psi = 0).$$

Denote by γ the minimax total error probability over $Q(\alpha, L, \varphi)$

$$\gamma := \gamma(Q(\alpha, L, \varphi)) = \inf_{\psi} \gamma(\psi, Q(\alpha, L, \varphi)).$$

- Our goal is to describe $\widetilde{\varphi} = \widetilde{\varphi}(n, p)$, called the separation rate, such that, on the one hand,

$$\gamma \longrightarrow 1 \quad \text{if } \frac{\varphi}{\widetilde{\varphi}} \longrightarrow 0$$

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- On the other hand, there exists a test ψ such that, its total error probability tends to 0

$$\gamma(\psi, Q(\alpha, L, \varphi)) \longrightarrow 0 \quad \text{if } \frac{\varphi}{\widetilde{\varphi}} \longrightarrow +\infty$$

and we say that ψ is a consistent test procedure. Therefore we can distinguish between the two hypotheses.
Note that in the following the asymptotic are taken when $p \to \infty$ and $n \to \infty$.

Testing the covariance matrix has been studied by several methods in the case of high-dimensional settings:

- The largest eigenvalue (Johnstone 2001 *Ann. Stat.*), (Berthet and Rigollet 2013 *Ann. Stat.*).

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- Maximum deviation (Xiao and Wu 2011 *arXiv*)

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- Modifying the following statistic :

$tr(S_n - Id)^2,$ where S_n is the is the sample covariance matrix

- Ledoit and Wolf (2002, *Ann. Stat.*) proposed the following test statistic:

$$W_n = \frac{1}{p} tr(S_n - Id)^2 - \frac{p}{n}\left(\frac{tr(S_n)}{p}\right)^2 + \frac{p}{n}$$

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- Chen and al. (2010 *JASA*) proposed a U-statistic defined as follows

$$U_n = \frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} (X_i^T X_j)^2 - \frac{2}{n} \sum_{i=1}^{p} X_i^T X_i + 1$$

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- The test statistic is the U-statistic U_n proposed by Chen and al. defined previously.
- The separation rate is $\widetilde{\varphi} = b\sqrt{p/n}$.

- Test statistic :

$$\widehat{\mathcal{D}}_n = \frac{1}{n(n-1)p} \sum_{\substack{l,k=1 \\ l \neq k}}^{n} \sum_{\substack{i,j=1 \\ i<j}}^{p} w_{ij} X_{k,i} X_{k,j} X_{l,i} X_{l,j} \qquad (1)$$

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- In order to define the weights $(w_{ij}^*)_{1 \leq i,j \leq p}$ that will appear in the optimal test procedure, and to distinguish the alternative from the null at the best, we have to resolve the following extremal problem:

$$\frac{1}{p} \sum_{\substack{i,j=1 \\ i<j}}^{p} w_{ij}^* \sigma_{ij}^{*2} = \sup_{\left\{ \begin{array}{l} (w_{ij})_{ij} \; : \; w_{ij} \geq 0; \\ \frac{1}{p} \sum_{\substack{i,j=1 \\ i<j}}^{p} w_{ij}^2 = 1 \end{array} \right\}} \inf_{\left\{ \begin{array}{l} \Sigma \; : \; \Sigma = (\sigma_{ij})_{i,j}; \\ \Sigma \in Q(\alpha, L, \varphi) \end{array} \right\}} \frac{1}{p} \sum_{\substack{i,j=1 \\ i<j}}^{p} w_{ij} \sigma_{ij}^2$$

The solutions of the optimization problem given above are:

- $w_{ij}^* = \frac{\lambda}{2b(\varphi)}\left(1 - \left(\frac{|i-j|}{T}\right)^{2\alpha}\right), \quad \sigma_{ij}^{*2} = \lambda\left(1 - (\frac{|i-j|}{T})^{2\alpha}\right)_+$

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- $T = \lfloor C_T(\alpha, L) \cdot \varphi^{-\frac{1}{\alpha}} \rfloor, \quad \lambda = C_\lambda(\alpha, L) \cdot \varphi^{\frac{2\alpha+1}{\alpha}}$

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- $w_{ij}^* \geq 0, \quad \dfrac{1}{p} \sum_{\substack{i,j=1 \\ i<j}}^{p} w_{ij}^{*2} = \dfrac{1}{2}$ and $\sup_{i,j} w_{ij}^* \asymp \dfrac{1}{\sqrt{T}}$.

Proposition

Assume $\varphi \longrightarrow 0$, $\alpha > 1$. Under the null hypothesis:

$$\mathbb{E}_{Id}(\widehat{\mathcal{D}}_n) = 0 \ , \quad Var_{Id}(\widehat{\mathcal{D}}_n) = \frac{1}{n(n-1)p}.$$

Under the alternative, for all $\Sigma \in Q(\alpha, L, \varphi)$,

$$\mathbb{E}_{\Sigma}(\widehat{\mathcal{D}}_n) = \frac{1}{p}\sum_{\substack{i,j=1\\ i<j}}^{p} w_{ij}^{*}\sigma_{ij}^{2} \geq b(\varphi) \ , \quad Var_{\Sigma}(\widehat{\mathcal{D}}_n) = \frac{T_1}{n(n-1)p^2} + \frac{T_2}{np^2}$$

where,

$$T_1 \leq \ p(1+o(1)) + c_1 \cdot T\sqrt{T} \cdot p \cdot \mathbb{E}_{\Sigma}(\widehat{\mathcal{D}}_n) + c_2 \cdot p^2 \cdot \mathbb{E}_{\Sigma}^{2}(\widehat{\mathcal{D}}_n)$$

$$T_2 \leq p \cdot \mathbb{E}_{\Sigma}(\widehat{\mathcal{D}}_n) \cdot o(1) + p \cdot \sqrt{p} \cdot \mathbb{E}_{\Sigma}(\widehat{\mathcal{D}}_n) \cdot o(1) + c_3 \cdot p^2 \cdot \mathbb{E}_{\Sigma}^{2}(\widehat{\mathcal{D}}_n)$$

We define the following test procedure

$$\psi^* = I(\widehat{\mathcal{D}}_n > t), \quad t > 0 \tag{2}$$

Theorem (Upper bound)

Under the asymptotic conditions, additionally assume that $\varphi \longrightarrow 0$, *The test procedure* ψ^* *defined in (4) with* $t > 0$ *has the following properties :*
Type I error probability : if $n\sqrt{p}t \longrightarrow +\infty$ *then* $\eta(\psi^*) \longrightarrow 0$.
Maximal type II error probability : if

$$n^2 p \varphi^{\frac{4\alpha+1}{\alpha}} C(\alpha, L) \longrightarrow +\infty \tag{3}$$

choose t *such that* $t \leq c \cdot \varphi^{\frac{4\alpha+1}{2\alpha}} C^{1/2}(\alpha, L)$, *for some constant* c ; $0 < c < 1$, *then* $\beta(\psi^*, Q(\alpha, L, \varphi)) \longrightarrow 0$.

Theorem (Lower bound)

Under the asymptotic conditions, additionally assume that $\alpha > 1/2$, *if* φ *is such that*

$$n^2 p \varphi^{\frac{4\alpha+1}{\alpha}} C(\alpha, L) \longrightarrow 0$$

then for any test ψ, *we have* $\gamma(\psi, Q(\alpha, L, \varphi)) \longrightarrow 1$, *which implies that*

$$\gamma = \inf_{\psi} \gamma(\psi, Q(\alpha, L, \varphi)) \longrightarrow 1.$$

As consequence of the previous theorems $\widetilde{\varphi} = (C(\alpha, L) n^2 p)^{-\frac{\alpha}{4\alpha+1}}$ is the sharp minimax separation rate.

Remark : $n, p \longrightarrow +\infty$ without restrictions.

- For the particular case when Σ is Toeplitz, we define the following class under the alternative:

$$Q'(\alpha, L\phi) = \{\Sigma \in \mathcal{T}(\alpha, L); \sum_{k=1}^{p} \sigma_k^2 \geq \phi^2\}$$

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- Test statistic :

$$\widehat{\mathcal{A}}_n = \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \sum_{k=1}^{T} \frac{w_k^*}{(p-T)^2} \sum_{l_1=T+1}^{p} \sum_{l_2=T+1}^{p} X_{i,l_1} X_{i,l_1-k} X_{j,l_2} X_{j,l_2-k}$$

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- Note that the previous test statistic is different from the one proposed by Ermakov(1994) and uses the independent copies of the stationary process that we observe.

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- We obtain sharp minimax rates for testing

$$\widetilde{\varphi} = (C(\alpha, L)n^2p^2)^{-\frac{\alpha}{4\alpha+1}} \text{ and } \widetilde{\varphi} = (C(\alpha, L)n^2p)^{-\frac{\alpha}{4\alpha+1}}$$

for Σ Toeplitz and non-Toeplitz, respectively. The additional factor p is due to the number of unknown parameters.

Thank you for your attention