Ninomiya-Victoir scheme: strong convergence, antithetic version and application to multilevel estimators

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April 18, 2016

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Outline

Goal

- We are interested in the computation, by Monte Carlo methods, of the expectation $Y = \mathbb{E}[f(X_T)]$, where $X = (X_t)_{t \in [0,T]}$ is the solution to a multidimensional stochastic differential equation (SDE) and $f : \mathbb{R}^n \mapsto \mathbb{R}$ a given function such that $\mathbb{E}\left[f(X_T)^2\right] < +\infty$.
- We will focus on minimizing the computational complexity subject to a given target error $\epsilon \in \mathbb{R}_+^*$.
- To measure the accuracy of an estimator $\hat{Y}$, we will consider the root mean squared error:

$$RMSE\left(\hat{Y}; Y\right) = \mathbb{E}^{\frac{1}{2}}\left[\left|Y - \hat{Y}\right|^2\right]. \tag{1}$$

Itô-type SDE

We consider a general Itô-type SDE of the form

$$\begin{cases} dX_t = b(X_t)dt + \sum_{j=1}^{d} \sigma^j(X_t)dW_t^j \\ X_0 = x \end{cases} \tag{2}$$

where:

- $x \in \mathbb{R}^n$,
- $(X_t)_{t\in[0,T]}$ is a $n-$dimensional stochastic process,
- $W = \left(W^1, \dots, W^d\right)$ is a $d-$dimensional standard Brownian motion,
- $b, \sigma^1, \dots, \sigma^d : \mathbb{R}^n \to \mathbb{R}^n$ are Lipschitz continuous.

Outline

Standard Monte Carlo Method

The standard Monte Carlo method consists in:

- discretizing the SDE, using a numerical scheme X^N, with $N \in \mathbb{N}^*$ steps,
- approximating the expectation using $M \in \mathbb{N}^*$ independent path simulations.

To be clear, the crude Monte Carlo estimator is given by

$$\hat{Y}_{CMC} = \frac{1}{M} \sum_{k=1}^{M} f\left(X_T^{N,k}\right) \tag{3}$$

where $X^{N,k}$ are independent copies of a numerical scheme X^N.

Complexity analysis

Bias

$$B\left(\hat{Y}_{CMC}; Y\right) = \mathbb{E}\left[\hat{Y}_{CMC}\right] - Y = \mathbb{E}\left[f\left(X_T^N\right)\right] - \mathbb{E}\left[f\left(X_T\right)\right]. \quad (4)$$

The bias is related to the weak error of the scheme:

$$\mathbb{E}\left[f\left(X_T^N\right) - f\left(X_T\right)\right] = \frac{c_1}{N^\alpha} + o\left(\frac{1}{N^\alpha}\right). \quad (5)$$

Variance

$$\mathbb{V}\left[\hat{Y}_{CMC}\right] = \frac{1}{M}\mathbb{V}\left[f\left(X_T^N\right)\right]. \quad (6)$$

Cost

$$\mathcal{C}_{CMC} = C \times M \times N = O\left(\epsilon^{-\left(2+\frac{1}{\alpha}\right)}\right). \quad (7)$$

Outline

The Multilevel Monte Carlo

The main idea of this technique is to use the following telescopic summation to control the bias:

$$\mathbb{E}\left[f\left(X_T^{2^L}\right)\right] = \mathbb{E}\left[f\left(X_T^1\right)\right] + \sum_{l=1}^{L} \mathbb{E}\left[f\left(X_T^{2^l}\right) - f\left(X_T^{2^{l-1}}\right)\right].$$

Then, a generalized multilevel Monte Carlo estimator is built as follows:

$$\hat{Y}_{MLMC} = \sum_{l=0}^{L} \frac{1}{M_l} \sum_{k=1}^{M_l} Z_k^l \tag{8}$$

where $\left(Z_k^l\right)_{0 \leq l \leq L, 1 \leq k \leq M_l}$ are independent random variables such that:

$$\mathbb{E}\left[Z^0\right] = \mathbb{E}\left[f\left(X_T^1\right)\right] \tag{9}$$

and:

$$\forall l \in \{1, \ldots, L\}, \mathbb{E}\left[Z^l\right] = \mathbb{E}\left[f\left(X_T^{2^l}\right) - f\left(X_T^{2^{l-1}}\right)\right]. \tag{10}$$

Bias and variance

Bias

$$B\left(\hat{Y}_{MLMC}; Y\right) = \mathbb{E}\left[\hat{Y}_{MLMC}\right] - Y = \mathbb{E}\left[f\left(X_T^{2^L}\right)\right] - \mathbb{E}\left[f\left(X_T\right)\right]. \quad (11)$$

The bias is related to the weak error of the scheme:

$$\mathbb{E}\left[f\left(X_T^{2^L}\right) - f\left(X_T\right)\right] = \frac{c_1}{2^{\alpha L}} + o\left(\frac{1}{2^{\alpha L}}\right). \quad (12)$$

Variance

$$\mathbb{V}\left[\hat{Y}_{MLMC}\right] = \sum_{l=0}^{L} \frac{1}{M_l}\mathbb{V}\left[Z^l\right]. \quad (13)$$

Cost and canonical exemple

Cost

For a given discretization level $l \in \{0, \ldots, L\}$, the computational cost of simulating one sample Z^l is $C\lambda_l 2^l$, where:

- $C \in \mathbb{R}_+$ is a constant, depending only on the discretization scheme,
- $\forall l \in \mathbb{N}, \lambda_l \in \mathbb{Q}_+^*$ is a weight, depending only on l,

$$\mathcal{C}_{MLMC} = C \sum_{l=0}^{L} M_l \lambda_l 2^l. \tag{14}$$

Natural choice for $Z^l, l \in \{0, \ldots, L\}$

$$Z^0 = f\left(X_T^1\right) \tag{15}$$

$$Z^l = f\left(X_T^{2^l}\right) - f\left(X_T^{2^{l-1}}\right), \forall l \in \{1, \ldots, L\}. \tag{16}$$

For this canonical choice, it is natural to take $\lambda_0 = 1$ and $\lambda_l = \frac{3}{2}$.

Optimal complexity

Theorem (Complexity theorem (Giles))

Assume that $\exists\,(\alpha, c_1) \in \mathbb{R}_+^* \times \mathbb{R}^*$ *and* $\exists\,(\beta, c_2) \in \left(\mathbb{R}_+^*\right)^2$ *such that* $\forall l \in \mathbb{N}$*:*

$$\mathbb{E}\left[f\left(X_T^{2^l}\right)\right] - Y = \frac{c_1}{2^{\alpha l}} + o\left(\frac{1}{2^{\alpha l}}\right) \tag{17}$$

and

$$\mathbb{V}\left[Z^l\right] = \frac{c_2}{2^{\beta l}} + o\left(\frac{1}{2^{\beta l}}\right). \tag{18}$$

Then, the optimal complexity is given by:

$$\begin{cases} \mathcal{C}^*_{MLMC} = O\left(\epsilon^{-2}\right) \ \ \textit{if } \beta > 1, \\ \mathcal{C}^*_{MLMC} = O\left(\epsilon^{-2}\left(\log\left(\frac{1}{\epsilon}\right)\right)^2\right) \ \ \textit{if } \beta = 1, \\ \mathcal{C}^*_{MLMC} = O\left(\epsilon^{-2+\frac{\beta-1}{\alpha}}\right) \ \ \textit{if } \beta < 1. \end{cases} \tag{19}$$

Optimal parameters

Optimal parameters

$$L^* = \left\lceil \frac{\log_2\left(\frac{\sqrt{2}|c_1|}{\epsilon}\right)}{\alpha} \right\rceil \tag{20}$$

$$M_0^* = \left\lceil \frac{2}{\epsilon^2}\sqrt{\frac{\mathbb{V}\left[Z^0\right]}{\lambda_0}}\left(\sqrt{\lambda_0 \mathbb{V}\left[Z^0\right]} + \sum_{l=1}^{L^*}\sqrt{c_2\lambda_l 2^{l(1-\beta)}}\right)\right\rceil \tag{21}$$

$$\forall l \in \{1,\ldots,L^*\}, M_l^* = \left\lceil \frac{2}{\epsilon^2}\sqrt{\frac{c_2}{\lambda_l 2^{l(\beta+1)}}}\left(\sqrt{\lambda_0 \mathbb{V}\left[Z^0\right]} + \sum_{l=1}^{L^*}\sqrt{c_2\lambda_l 2^{l(1-\beta)}}\right)\right\rceil . \tag{22}$$

Regression

One can estimate $(\alpha, \beta, c_1, c_2)$ by using a regression:

$$\mathbb{V}\left[Z^l\right] \sim \frac{c_2}{2^{\beta l}} \tag{23}$$

$$\mathbb{E}\left[Z^l\right] \sim \frac{c_1\left(1-2^{\alpha}\right)}{2^{\alpha l}}. \tag{24}$$

Outline

Stratonovich form

Assuming $\mathcal{C}^1$ regularity for diffusion coefficients $\sigma^1, \ldots, \sigma^d$, the Itô-type SDE can be written in Stratonovich form:

$$\begin{cases} dX_t = \sigma^0(X_t)dt + \sum_{j=1}^{d} \sigma^j(X_t) \circ dW_t^j \\ X_0 = x \end{cases} \tag{25}$$

where $\sigma^0 = b - \frac{1}{2} \sum_{j=1}^{d} \partial\sigma^j \sigma^j$ and $\partial\sigma^j$ is the Jacobian matrix of σ^j defined as follows

$$\partial\sigma^j = \left(\left(\partial\sigma^j\right)_{ik} \right)_{i,k \in [\![1;n]\!]} = \left(\partial_{x_k} \sigma^{ij}\right)_{i,k \in [\![1;n]\!]}. \tag{26}$$

The Ninomiya-Victoir scheme

Notations

- $\left(t_k = k\frac{T}{N}\right)_{k\in[\![0;N]\!]}$ is the subdivision of $[0, T]$.
- $\eta^N = (\eta_1, \ldots, \eta_N)$ is a sequence of independent, identically distributed Rademacher random variables independent of W.
- $\forall j \in \{1, \ldots, d\}, \Delta W^j_{t_{k+1}} = W^j_{t_{k+1}} - W^j_{t_k}$.
- For $j \in \{0, \ldots, d\}$ and $x_0 \in \mathbb{R}^d$, let $(\exp(t\sigma^j)x_0)_{t\in\mathbb{R}}$ solve the ODE

$$\begin{cases} \frac{dx(t)}{dt} = \sigma^j(x(t)) \\ x(0) = x_0. \end{cases}$$

Scheme

If $\eta_{k+1} = 1$

$$X^{NV,N,\eta^N}_{t_{k+1}} = \exp\left(\frac{T}{2N}\sigma^0\right)\exp\left(\Delta W^d_{t_{k+1}}\sigma^d\right)\ldots\exp\left(\Delta W^1_{t_{k+1}}\sigma^1\right)\exp\left(\frac{T}{2N}\sigma^0\right)X^{NV,N,\eta^N}_{t_k}$$

and if $\eta_{k+1} = -1$

$$X^{NV,N,\eta^N}_{t_{k+1}} = \exp\left(\frac{T}{2N}\sigma^0\right)\exp\left(\Delta W^1_{t_{k+1}}\sigma^d\right)\ldots\exp\left(\Delta W^d_{t_{k+1}}\sigma^1\right)\exp\left(\frac{T}{2N}\sigma^0\right)X^{NV,N,\eta^N}_{t_k}.$$

Order 2 of weak convergence

Denoting by $(X_t^x)_{t\geq 0}$ the solution to the SDE starting from $X_0^x = x \in \mathbb{R}^n$, for $f : \mathbb{R}^n \to \mathbb{R}^n$ smooth, $u(t,x) = \mathbb{E}\left[f(X_t^x)\right]$ solves the Feynman-Kac PDE

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = Lu(t,x), \ (t,x) \in [0,\infty) \times \mathbb{R}^n \\ u(0,x) = f(x), \ x \in \mathbb{R}^n \end{cases}$$

with $L = b.\nabla_x + \frac{1}{2}\mathrm{Tr}\left[(\sigma^1,\dots,\sigma^d)(\sigma^1,\dots,\sigma^d)^*\nabla_x^2\right] = \sigma^0 + \frac{1}{2}\sum_{j=1}^d (\sigma^j)^2$ the infinitesimal generator.

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t} Lu = L\frac{\partial}{\partial t} u = L^2 u$$

$$\text{and } u(t_1,x) = f(x) + t_1 Lf(x) + \frac{t_1^2}{2}L^2 f(x) + \mathcal{O}(t_1^3).$$

Ninomiya and Victoir have designed their scheme so that

$$\mathbb{E}[f(X_{t_1}^{NV,N,\eta^N})] = f(x) + t_1 Lf(x) + \frac{t_1^2}{2}L^2 f(x) + \mathcal{O}(t_1^3).$$

One step error $\mathcal{O}(\frac{1}{N^3}) \overset{N\text{steps}}{\longrightarrow} \mathcal{O}(\frac{1}{N^2})$ global error.

Order 1/2 of strong convergence

Theorem (Strong convergence)

Assume that the vector fields, $b, \forall j \in \{1, \ldots, d\}, \sigma^j$ *and* $\partial\sigma^j\sigma^j$ *are Lipschitz continuous functions. Then:* $\forall p \geq 1, \exists C_{NV} \in \mathbb{R}_+^*, \forall N \in \mathbb{N}^*$

$$\mathbb{E}\left[\max_{0\leq k\leq N}\left\|X_{t_k} - X_{t_k}^{NV,N,\eta^N}\right\|^{2p}\middle|\eta\right] \leq \frac{C_{NV}}{N^p}. \tag{27}$$

Outline

The Ninomiya-Victoir scheme: antithetic version

We consider two grids: a coarse grid with time step $h_{l-1} = \frac{T}{2^{l-1}}$, a fine grid with time step $h_l = \frac{T}{2^l}$ and we introduce some notations:

- $\forall k \in \left\{0, \ldots, 2^{l-1}\right\}, t_k = kh_{l-1}$,
- $\forall k \in \left\{0, \ldots, 2^{l-1} - 1\right\}, t_{k+\frac{1}{2}} = \left(k + \frac{1}{2}\right) h_{l-1}$,
- $\eta^{2^l} = (\eta_1, \ldots, \eta_{2^l})$,
- $\Delta W^c_{t_{k+1}} = W_{t_{k+1}} - W_{t_k}$, $\Delta W^f_{t_{k+\frac{1}{2}}} = W_{t_{k+\frac{1}{2}}} - W_{t_k}$ and $\Delta W^f_{t_{k+1}} = W_{t_{k+1}} - W_{t_{k+\frac{1}{2}}}$.

On the coarsest grid, $X^{NV,2^{l-1},\eta^{2^l}}$ is defined inductively by: $\eta_{2k+1} = 1$:

$$X^{NV,2^{l-1},\eta^{2^l}}_{t_{k+1}} = \exp\left(\frac{h_{l-1}}{2}\sigma^0\right) \exp\left(\Delta W^{d,c}_{t_{k+1}}\sigma^d\right) \ldots \exp\left(\Delta W^{1,c}_{t_{k+1}}\sigma^1\right) \exp\left(\frac{h_{l-1}}{2}\sigma^0\right) X^{NV,2^{l-1},\eta^{2^l}}_{t_k}, \tag{28}$$

and if $\eta_{2k+1} = -1$:

$$X^{NV,2^{l-1},\eta^{2^l}}_{t_{k+1}} = \exp\left(\frac{h_{l-1}}{2}\sigma^0\right) \exp\left(\Delta W^{1,c}_{t_{k+1}}\sigma^d\right) \ldots \exp\left(\Delta W^{d,c}_{t_{k+1}}\sigma^1\right) \exp\left(\frac{h_{l-1}}{2}\sigma^0\right) X^{NV,2^{l-1},\eta^{2^l}}_{t_k}. \tag{29}$$

The Ninomiya-Victoir: antithetic version

Similarly, on the finest grid: $\eta_{2k+1} = 1$:

$$X^{NV,2^l,\eta^{2^l}}_{t_{k+\frac{1}{2}}} = \exp\left(\frac{h_l}{2}\sigma^0\right) \exp\left(\Delta W^{d,f}_{t_{k+\frac{1}{2}}} \sigma^d\right) \dots \exp\left(\Delta W^{1,f}_{t_{k+\frac{1}{2}}} \sigma^1\right) \exp\left(\frac{h_l}{2}\sigma^0\right) X^{NV,2^l,\eta^{2^l}}_{t_k}, \tag{30}$$

and if $\eta_{2k+1} = -1$:

$$X^{NV,2^l,\eta^{2^l}}_{t_{k+\frac{1}{2}}} = \exp\left(\frac{h_l}{2}\sigma^0\right) \exp\left(\Delta W^{1,f}_{t_{k+\frac{1}{2}}} \sigma^d\right) \dots \exp\left(\Delta W^{d,f}_{t_{k+\frac{1}{2}}} \sigma^1\right) \exp\left(\frac{h_l}{2}\sigma^0\right) X^{NV,2^l,\eta^{2^l}}_{t_k}, \tag{31}$$

if $\eta_{2k+2} = 1$:

$$X^{NV,2^l,\eta^{2^l}}_{t_{k+1}} = \exp\left(\frac{h_l}{2}\sigma^0\right) \exp\left(\Delta W^{d,f}_{t_{k+1}} \sigma^d\right) \dots \exp\left(\Delta W^{1,f}_{t_{k+1}} \sigma^1\right) \exp\left(\frac{h_l}{2}\sigma^0\right) X^{NV,2^l,\eta^{2^l}}_{t_{k+\frac{1}{2}}}, \tag{32}$$

and if $\eta_{2k+2} = -1$:

$$X^{NV,2^l,\eta^{2^l}}_{t_{k+1}} = \exp\left(\frac{h_l}{2}\sigma^0\right) \exp\left(\Delta W^{1,f}_{t_{k+1}} \sigma^d\right) \dots \exp\left(\Delta W^{d,f}_{t_{k+1}} \sigma^1\right) \exp\left(\frac{h_l}{2}\sigma^0\right) X^{NV,2^l,\eta^{2^l}}_{t_{k+\frac{1}{2}}}. \tag{33}$$

The antithetic scheme $\tilde{X}^{NV,2^l,\eta^{2^l}}$ is defined by the same discretization, except that the Brownian increment $\Delta W^f_{t_{k+\frac{1}{2}}}$ and $\Delta W^f_{t_{k+1}}$ are swapped.

Strong coupling with order one between successive levels

Considering:

$$Z_{NV}^{l} = \frac{1}{4}\left(f\left(\tilde{X}_{T}^{NV,2^{l},\eta^{2^{l}}}\right) + f\left(\tilde{X}_{T}^{NV,2^{l},-\eta^{2^{l}}}\right) + f\left(X_{T}^{NV,2^{l},\eta^{2^{l}}}\right) + f\left(X_{T}^{NV,2^{l},-\eta^{2^{l}}}\right)\right) - \frac{1}{2}\left(f\left(X_{T}^{NV,2^{l-1},\eta^{2^{l}}}\right) + f\left(X_{T}^{NV,2^{l-1},-\eta^{2^{l}}}\right)\right), \quad (34)$$

we have a first order of convergence.

Theorem

Assume that $f \in \mathcal{C}^{2}(\mathbb{R}^{n},\mathbb{R})$ and $b \in \mathcal{C}^{2}(\mathbb{R}^{n},\mathbb{R}^{n})$ with bounded first and second order derivatives, and, $\forall j \in \{1,\ldots,d\}, \sigma^{j} \in \mathcal{C}^{3}(\mathbb{R}^{n},\mathbb{R}^{n})$ with bounded first and second order derivatives and with polynomially growing third order derivatives. Then:

$$\forall p \geq 1, \exists c \in \mathbb{R}_{+}^{*}, \forall l \in \mathbb{N}^{*}, \ \mathbb{E}\left[\left|Z_{NV}^{l}\right|^{2p}\right] \leq \frac{c}{2^{2pl}}. \quad (35)$$

Derived MLMC estimator

The antithetic MLMC estimator, $\hat{Y}^{NV}_{MLMC}$, with the Ninomiya-Victoir scheme is defined as follows

$$\hat{Y}^{NV}_{MLMC} = \sum_{l=0}^{L^*} \frac{1}{M_l^*} \sum_{k=1}^{M_l^*} Z^{l,k}_{NV}$$

where $Z^0_{NV} = f\left(X_T^{NV,1,\eta}\right)$ or $Z^0_{NV} = \frac{1}{2}\left(f\left(X_T^{NV,1,\eta}\right) + f\left(X_T^{NV,1,-\eta}\right)\right)$, and for $l \in \{0, \ldots, L^*\}$,$Z^{l,k}_{NV}$ are independent copies of Z^l_{NV}.

Practical procedure

- Step 1: Estimate α, β, c_1, c_2 and $\mathbb{V}\left[Z^0_{NV}\right]$.
- Step 2: Compute L^* and $(M_l^*)_{0 \le l \le L^*}$.
- Step 3: Compute $\hat{Y}^{NV}_{MLMC}$.

Outline

The Heston model

Heston model

$$\begin{cases} dU_t = (r - \delta - \frac{1}{2}V_t)dt + \sqrt{V_t}dW_t^1 \\ dV_t = \kappa(\theta - V_t)dt + \sigma\sqrt{V_t}\left(\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2\right), \end{cases} \quad (36)$$

where the asset price S is given by $S_t = \exp(U_t)$ and

- $\theta \in \mathbb{R}_+^*$ is the long implied variance, or long run average price variance; as t tends to infinity, the expected value of V_t tends to θ,
- $\kappa \in \mathbb{R}_+^*$ is the rate at which V_t reverts to θ,
- $\sigma \in \mathbb{R}_+^*$ is the volatility of the implied volatility and determines the variance of V_t,
- $r \in \mathbb{R}$ the annualized risk-free interest rate, continuously compounded,
- $\delta \in \mathbb{R}_+^*$ is the annualized continuous yield dividend,
- $\rho \in]-1, 1[$ is the correlation between the two Brownian motion (ie stock price and implied volatility).

The Heston model

In this 2−dimensional model, the Brownian vector fields are given by

- $\sigma^1 \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \sqrt{v} \\ \rho\sigma\sqrt{v} \end{pmatrix}$,
- $\sigma^2 \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ \sigma\sqrt{1-\rho^2}\sqrt{v} \end{pmatrix}$.

The drift coefficient is $b \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} r - \delta - \frac{1}{2}v \\ \kappa(\theta - v) \end{pmatrix}$.

The Stratonovich drift is given by $\sigma^0 = b - \frac{1}{2}\left(\partial\sigma^1\sigma^1 + \partial\sigma^2\sigma^2\right)$:

$$\sigma^0 \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} r - \delta - \frac{1}{2}v - \frac{1}{4}\rho\sigma \\ \kappa(\theta - v) - \frac{\sigma^2}{4} \end{pmatrix}.$$

Outline

The Ninomiya-Victoir scheme

The Ninomiya-Victoir scheme is well defined fo setting $\xi = \theta - \frac{\sigma^2}{4\kappa} \geq 0$, for a given uniform grid, is inductively defined by:

1st step:

$$\bar{U}^0_{t_{k+1}} = U^{NV,\eta}_{t_k} + \frac{1}{2}\left(r - \delta - \frac{1}{2}\rho\sigma - \frac{1}{2}\xi\right) t_1 + \frac{1}{2\kappa}\left(v - \xi\right)\left(\exp\left(-\frac{1}{2}\kappa t_1\right) - 1\right),$$

$$\bar{V}^0_{t_{k+1}} = \left(V^{NV,\eta}_{t_k} - \xi\right)\left(\exp\left(-\frac{1}{2}\kappa t_1\right) - 1\right) + \xi.$$

2nd step:

If $\eta_{k+1} = 1$:

$$\bar{U}^{1,\eta}_{t_{k+1}} = \bar{U}^0_{t_{k+1}} + \sqrt{\bar{V}^0_{t_{k+1}}}\Delta W^1_{t_{k+1}} + \frac{1}{4}\rho\sigma\left(\Delta W^1_{t_{k+1}}\right)^2,$$

$$\bar{V}^{1,\eta}_{t_{k+1}} = \left(\sqrt{\bar{V}^0_{t_{k+1}}} + \frac{1}{2}\sigma\rho\Delta W^1_{t_{k+1}}\right)^2,$$

$$\bar{U}^{2,\eta}_{t_{k+1}} = \bar{U}^{1,\eta}_{t_{k+1}},$$

$$\bar{V}^{2,\eta}_{t_{k+1}} = \left(\sqrt{\bar{V}^{1,\eta}_{t_{k+1}}} + \frac{1}{2}\sigma\sqrt{1-\rho^2}\Delta W^2_{t_{k+1}}\right)^2.$$

If $\eta_{k+1} = -1$:

$$\bar{U}^{1,\eta}_{t_{k+1}} = \bar{U}^{0,\eta}_{t_{k+1}},$$

$$\bar{V}^{1,\eta}_{t_{k+1}} = \left(\sqrt{\bar{V}^{0}_{t_{k+1}}} + \frac{1}{2}\sigma\sqrt{1-\rho^2}\Delta W^2_{t_{k+1}} \right)^2,$$

$$\bar{U}^{2,\eta}_{t_{k+1}} = \bar{U}^{1,\eta}_{t_{k+1}} + \sqrt{\bar{V}^{1,\eta}_{t_{k+1}}}\Delta W^1_{t_{k+1}} + \frac{1}{4}\rho\sigma\left(\Delta W^1_{t_{k+1}}\right)^2,$$

$$\bar{V}^{2,\eta}_{t_{k+1}} = \left(\sqrt{\bar{V}^{1,\eta}_{t_{k+1}}} + \frac{1}{2}\sigma\rho\Delta W^1_{t_{k+1}} \right)^2.$$

3rd step:

$$U^{NV,\eta}_{t_{k+1}} = \bar{U}^{2,\eta}_{t_{k+1}} + \frac{1}{2}\left(r - \delta - \frac{1}{2}\rho\sigma - \frac{1}{2}\xi\right)t_1 + \frac{1}{2\kappa}\left(v-\xi\right)\left(\exp\left(-\frac{1}{2}\kappa t_1\right) - 1\right),$$

$$V^{NV,\eta}_{t_{k+1}} = \left(\bar{V}^{2,\eta}_{t_{k+1}} - \xi\right)\left(\exp\left(-\frac{1}{2}\kappa t_1\right) - 1\right) + \xi.$$

The Ninomiya-Victoir scheme

If $\eta_{k+1} = -1$:

$$\bar{U}^{1,\eta}_{t_{k+1}} = \bar{U}^{0,\eta}_{t_{k+1}},$$

$$\bar{V}^{1,\eta}_{t_{k+1}} = \left(\sqrt{\bar{V}^{0}_{t_{k+1}}} + \frac{1}{2}\sigma\sqrt{1-\rho^2}\Delta W^2_{t_{k+1}} \right)^2,$$

$$\bar{U}^{2,\eta}_{t_{k+1}} = \bar{U}^{1,\eta}_{t_{k+1}} + \sqrt{\bar{V}^{1,\eta}_{t_{k+1}}}\Delta W^1_{t_{k+1}} + \frac{1}{4}\rho\sigma\left(\Delta W^1_{t_{k+1}}\right)^2,$$

$$\bar{V}^{2,\eta}_{t_{k+1}} = \left(\sqrt{\bar{V}^{1,\eta}_{t_{k+1}}} + \frac{1}{2}\sigma\rho\Delta W^1_{t_{k+1}} \right)^2.$$

3rd step:

$$U^{NV,\eta}_{t_{k+1}} = \bar{U}^{2,\eta}_{t_{k+1}} + \frac{1}{2}\left(r - \delta - \frac{1}{2}\rho\sigma - \frac{1}{2}\xi \right) t_1 + \frac{1}{2\kappa}(v-\xi)\left(\exp\left(-\frac{1}{2}\kappa t_1 \right) - 1 \right),$$

$$V^{NV,\eta}_{t_{k+1}} = \left(\bar{V}^{2,\eta}_{t_{k+1}} - \xi \right)\left(\exp\left(-\frac{1}{2}\kappa t_1 \right) - 1 \right) + \xi.$$

Advantages and drawbacks

Advantages

- This method is well defined when $4\kappa\theta \geq \sigma^2$.
- It is very efficient when the volatility process is far from 0.

Drawbacks

- The constants α, β, c_1 and c_2 are very sensitive to the parameters of the model.