Spiking neural models: from point processes to partial differential equations.

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Les Houches

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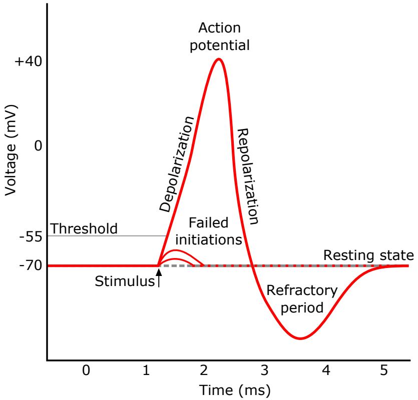
Biological context



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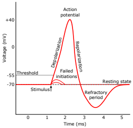
- Action potential = spike of the electrical potential.
- Physiological constraint: refractory period.

Biological context

microscopic scale



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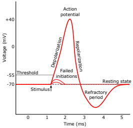

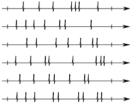

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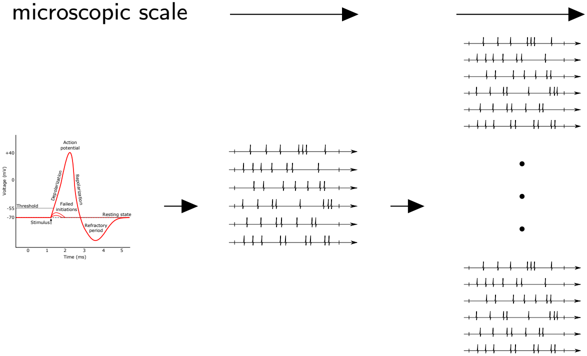

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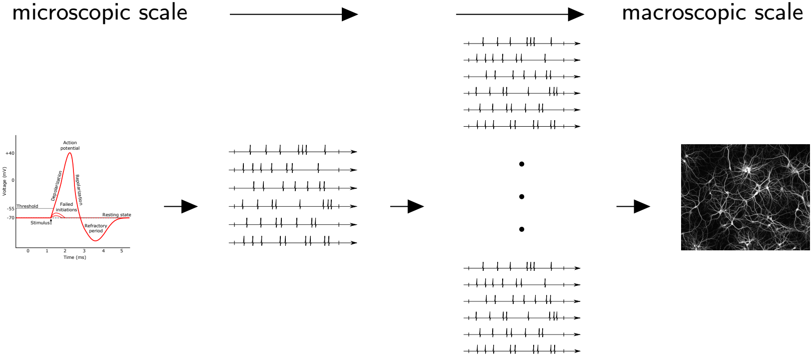

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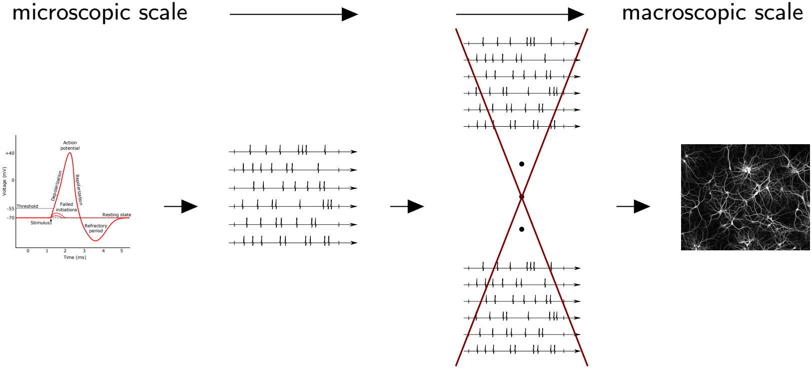

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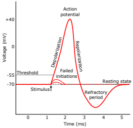

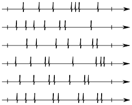

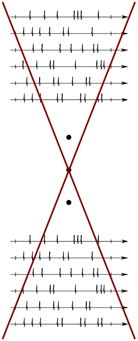

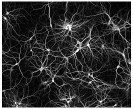

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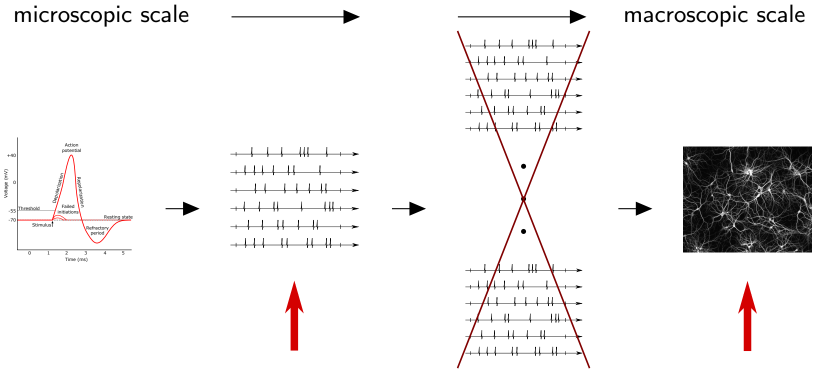

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Microscopic modelling of spike trains

Time point processes = random countable sets of times (points of $\mathbb{R}$ or $\mathbb{R}_+$).

- Point process: $N = \{T_i, i \in \mathbb{Z}\}$ s.t. $\cdots < T_0 \leq 0 < T_1 < \cdots$.
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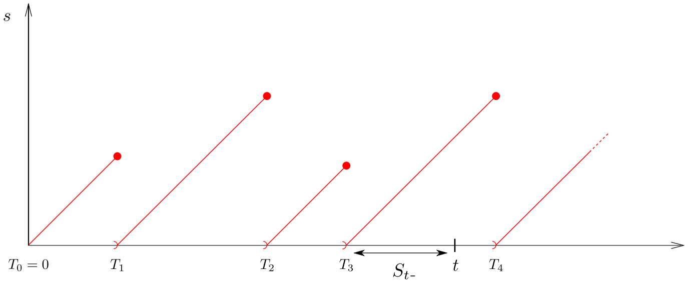

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Stochastic intensity

- Heuristically,

$$\lambda_t = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \mathbb{P}\left(N([t, t+\Delta t]) = 1 \,|\, \mathscr{F}^N_{t-} \right),$$

  where $\mathscr{F}^N_{t-}$ denotes the history of N before time t.
- Local behaviour: probability to find a new spike.
- May depend on the past (e.g. refractory period, excitation, inhibition).

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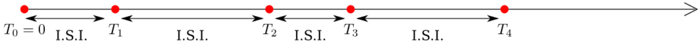

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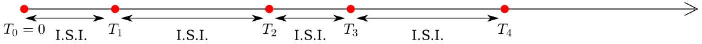

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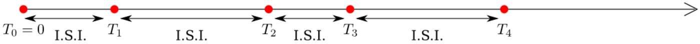

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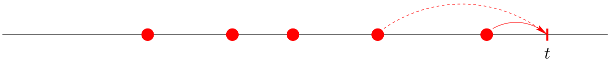

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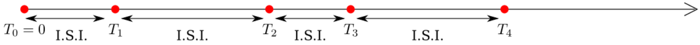

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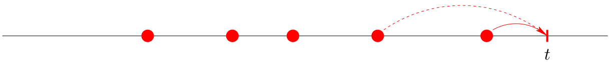

Model	Poisson	Renewal	Hawkes
Goodness-of-fit	×	✓	✓

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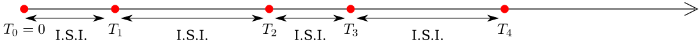

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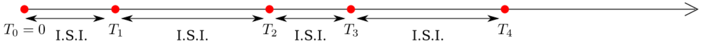

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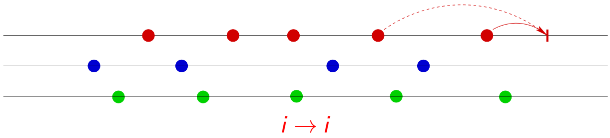

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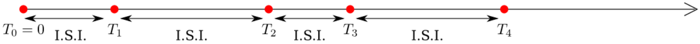

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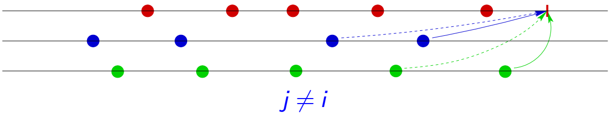

Age structured equations (K. Pakdaman, B. Perthame, D. Salort, 2010)

- Age $=$ delay since last spike.
- $u(t,s) = \begin{cases} \text{probability density of finding a neuron with age } s \text{ at time } t. \\ \text{ratio of the neural population with age } s \text{ at time } t. \end{cases}$

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Key Parameter

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- This system has been designed to describe a population of interacting neurons $\Rightarrow$ Mean-field theory.

Propagation of chaos: a tool to link the two scales

Mean field n-neurons system

- The neurons are dependent.
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- The dynamics is described by a growing system of equations.

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Age dependent Hawkes process (n-neurons system)

It is a multivariate point process $(N^i)_{i=1,..,n}$ with intensity given for all $i=1,\dots,n$ by

$$\lambda_t^i = \Psi\left(S_{t-}^i, \frac{1}{n}\sum_{j=1}^{n}\int_0^{t-} h(t-z)N^j(dz)\right). \quad \text{“}h_{j\to i} = \frac{1}{n}h\text{”}$$

- Example: $\Psi(s,x) = \Phi(x)\mathbb{1}_{s\geq\delta} \rightsquigarrow$ strict refractory period of length δ.
- How to approximate them as $n \to +\infty$?

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- The intensity of $\overline{N}$ depends on the time and the age.

Link between the limit process and the (PPS) system

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Proposition

If starting from a density, the distribution of the age $\overline{S}_{t-}$ admits a density denoted $u(t,\cdot)$ for all $t \geq 0$.
Moreover, u is the unique solution of the following (PPS) system

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- What about the real dynamics ?

From the n-neurons system to the PDE

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Fix k in $\mathbb{N}$. Then, the processes $N^1, \ldots, N^k$ of the n-neurons system behave at the limit when $n \to +\infty$ as i.i.d. copies of the limit process $\overline{N}$.

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Theorem

If the ages at time 0 are i.i.d. with common density u^{in}, then for all $t \geq 0$,

$$\frac{1}{n} \sum_{i=1}^{n} \delta_{S^i_{t-}} \xrightarrow[n \to \infty]{} u(t, \cdot),$$

where u is the unique solution of the (PPS) system with initial condition u^{in}.

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- Link between (PPS) and a well-designed microscopic model.
- Goodness-of fit tests: Renewal and Hawkes processes.

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Outlook:

- Highlight interesting dynamics at both scales.
- Fluctuations around the mean limit behaviour (Central Limit Theorem).
- Goodness of fit tests for both micro and macro models at the same time.