Estimation of the self-similarity and the stability indices through negative power variations

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Outline

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State of the art

- Self-similar processes are important in probability: connect to limit theorems, be of great interest in modeling, appear in geophysics, hydrology, turbulence, economics....
- Stable distributions are the only distributions that can be obtained as limits of normalized sums of i.i.d random variables.

State of the art

- Let $a = (a_0, \ldots, a_K)$, $K, L \in \mathbb{N}$ such that for $q = 0, \ldots, L$

$$\sum_{k=0}^{K} k^q a_k = 0, \sum_{k=0}^{K} k^{L+1} a_k \neq 0$$

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  e.g $K = 2, L = 1 : (a_0, a_1, a_2) = (-1, 2, -1)$.
- The increments of the process X with respect to a are defined by

$$\triangle_{p,n} X = \sum_{k=0}^{K} a_k X(\frac{k+p}{n}) \tag{1}$$

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- A usual statistical tool is the $\phi-$ variations:

$$V_n(\phi, X) = \frac{1}{n-K+1} \sum_{p=0}^{n-K} \phi(|\triangle_{p,n} X|)$$

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- Complex variations $\phi(x) = x^{iM}, M \in \mathbb{R}$ [Istas2012a].

State of the art

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▲ Objective: estimate both H and α, using β-variations, $\beta \in (-\frac{1}{2}, 0)$.

Outline

H-sssi process

A real-valued process X

- is *H-self-similar (H-ss)* if for all $a > 0$,

$$\{X(at), t \in \mathbb{R}\} \stackrel{(d)}{=} a^H \{X(t), t \in \mathbb{R}\},$$

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- has *stationary increments (si)* if, for all $s \in \mathbb{R}$,

$$\{X(t+s) - X(s), t \in \mathbb{R}\} \stackrel{(d)}{=} \{X(t) - X(0), t \in \mathbb{R}\}.$$

α-stable process

- A r.v X is said to have **a symmetric α-stable distribution ($S\alpha S$)** if there are parameters $0 < \alpha \leq 2, \sigma > 0$ such that its characteristic function has the following form:

$$\mathbb{E}e^{i\theta X} = \exp\left(-\sigma^{\alpha} \mid \theta \mid^{\alpha}\right)$$

We can write $X \sim S_{\alpha}(\sigma, 0, 0)$.

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- $\{X(t), t \in T\}$ is **symmetric stable** if all of its finite-dimensional distributions are symmetric stable.

Outline

1. Introduction
   - State of the art
   - Preliminary
2. Main results
   - H-sssi, $S\alpha S$-stable random processes
     - Settings and assumptions
     - Estimation of H and α
     - Examples
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     - Settings
     - Results and examples
   - Multifractional stable processes
3. Conclusion

Settings and assumptions

- Let X be a $H - sssi, S\alpha S$ random process ($\alpha \in (0, 2]$)
- The increments of X with respect to a are defined by

$$\triangle_{p,n} X = \sum_{k=0}^{K} a_k X(\frac{k+p}{n}) \qquad (2)$$

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- Let $\beta \in \mathbb{R}, -\frac{1}{2} < \beta < 0$, set

$$V_n(\beta) = \frac{1}{n-K+1} \sum_{p=0}^{n-K} |\triangle_{p,n}X|^{\beta} \tag{3}$$

$$W_n(\beta) = n^{\beta H} V_n(\beta) \tag{4}$$

$$\widehat{H_n} = \frac{1}{\beta} \log_2 \frac{V_{n/2}(\beta)}{V_n(\beta)} \tag{5}$$

An estimator of α

- Let $u, v \in \mathbb{R}$ such that $0 < v < u$.
- $g_{u,v} : (0, +\infty) \to \mathbb{R}$

$$g_{u,v}(x) = u \ln \left(\Gamma(1 + vx) \right) - v \ln \left(\Gamma(1 + ux) \right),$$

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- $h_{u,v} : (0, +\infty) \to (-\infty, 0)$

$$h_{u,v}(x) = g_{u,v}(1/x),$$

An estimator of α

- $\psi_{u,v} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$

$$\psi_{u,v}(x,y) = -v \ln x + u \ln y + C(u,v),$$

$$C(u,v) = \frac{u-v}{2} \ln \pi + u \ln \Gamma(1+v/2) + v \ln \Gamma(\frac{1-u}{2}) - v \ln \Gamma(1+u/2) - u \ln \Gamma(\frac{1-v}{2}),$$

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- $\varphi_{u,v} : \mathbb{R} \to [0,+\infty)$

$$\varphi_{u,v}(x) = \begin{cases} 0, & x \geq 0 \\ h_{u,v}^{-1}(x), & x < 0 \end{cases}$$

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$$\widehat{\alpha_n} = \varphi_{-\beta_1,-\beta_2}\left(\psi_{-\beta_1,-\beta_2}(V_n(\beta_1), V_n(\beta_2))\right)$$

Assumptions

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- $$\lim_{n\to\infty} \frac{1}{n} \sum_{p\in\mathbb{Z},|p|\leq n} |cov(|\triangle_{p,1}X|^{\beta}, |\triangle_{0,1}X|^{\beta})| = 0 \qquad (6)$$

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- There exists a sequence $\{b_n, n \in \mathbb{N}\}$, $\lim\limits_{n\to+\infty} b_n = 0$ such that

$$\limsup_{n\to\infty} \frac{1}{nb_n^2} \sum_{p\in\mathbb{Z},|p|\leq n} |cov(|\triangle_{p,1}X|^{\beta}, |\triangle_{0,1}X|^{\beta})| \leq C^2, \qquad (7)$$

where C is a constant.

Estimation of H and α

Theorem 1

1. Assume (6), then

$$\lim_{n\to+\infty} \widehat{H_n} \stackrel{(\mathbb{P})}{=} H, \lim_{n\to+\infty} \widehat{\alpha_n} \stackrel{(\mathbb{P})}{=} \alpha.$$

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2. *Assume (7), then*

$$\widehat{H_n} - H = O_{\mathbb{P}}(b_n), \widehat{\alpha_n} - \alpha = O_{\mathbb{P}}(b_n).$$

Examples

- **Well-balanced linear fractional stable motions**
  M: a $S\alpha S$ random measure $(0 < \alpha < 2)$ with Lebesgue control measure.

$$X(t) = \int_{\mathbb{R}} (|t-s|^{H-1/\alpha} - |s|^{H-1/\alpha}) M(ds)$$

with $H \in (0,1), H \neq 1/\alpha$.

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- **Takenaka's processes**
  $t \in \mathbb{R}$, set $C_t = \{(x,r) \in \mathbb{R} \times \mathbb{R}^+, |x-t| \leq r\}, S_t = C_t \triangle C_0$.
  M: a $S\alpha S$ random measure $(0 < \alpha < 2)$ with control measure

$$m(dx, dr) = r^{\nu-2} dxdr, (0 < \nu < 1).$$

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  The process X is ν/α-sssi.

Examples

Theorem 1 is true for

- well-balanced linear fractional stable motions, with

$$b_n = \begin{cases} n^{-1/2} & \text{, if } H < L + 1 - \frac{2}{\alpha} \\ n^{\frac{\alpha H - (L+1)\alpha}{4}} & \text{, if } H > L + 1 - \frac{2}{\alpha} \\ \sqrt{\frac{\ln n}{n}} & \text{, if } H = L + 1 - \frac{2}{\alpha} \end{cases} \tag{8}$$

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- Takenaka's processes, with

$$b_n = n^{\frac{\nu - 1}{2}}, \nu \in (0, 1) \tag{9}$$

CLT for fractional Brownian motions ($\alpha = 2$) and $S\alpha S$ Lévy motions

- **Fractional Brownian motion** is the unique, up to a constant, centered Gaussian H-sssi process, with $H \in (0, 1]$. Its covariance is given by

$$R(t, s) = \frac{C}{2}\{|s|^{2H} + |t|^{2H} - |s - t|^{2H}\}.$$

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- $\{X(t), t \geq 0\}$ with:
  - $X(0) = 0$ a.s,
  - has independent increments,
  - $X(t) - X(s) \sim S_\alpha((t-s)^{1/\alpha}, 0, 0)$ for any $0 \leq s < t < \infty$ and $0 < \alpha \leq 2$

  is called **a** $S\alpha S$ **Lévy motion**.

CLT for fractional Brownian motions ($\alpha = 2$), $S\alpha S$ Lévy motions

Theorem 2

Let X be a fBm (or $S\alpha S$-stable Lévy motion), then we have:

a)

$$\lim_{n\to+\infty} \widehat{H_n} \overset{\mathbb{P}}{=} H, \lim_{n\to+\infty} \widehat{\alpha_n} \overset{\mathbb{P}}{=} \alpha$$

b) $\sqrt{n}(\widehat{H_n} - H)$ *converges in distribution as* $n \to +\infty$*, to a centered Gaussian variable.*

c) $\sqrt{n}(\widehat{\alpha_n} - \alpha)$ *converges in distribution as* $n \to +\infty$*, to a centered Gaussian variable.*

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Settings

- $a = (a_0, \ldots, a_K)$, for $q = 0, \ldots, L$,

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- $k = (k_1, \ldots, k_d) \in \mathbb{N}^d$,

$$\triangle_{k,n} X = \sum_{p=(p_1,\ldots,p_d), p_i=0}^{K} a_p X\left(\frac{k+p}{n}\right)$$

Settings

- Fix $-1/2 < \beta < 0$, let

$$V_n(\beta) = \frac{1}{(n-K+1)^d} \sum_{k=(k_1,\dots,k_d),k_i=0}^{n-K} |\triangle_{k,n} X|^{\beta}$$

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Estimation of H and α

Asumptions:

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$$\lim_{n\to+\infty}\frac{1}{n^d}\sum_{k=(k_1,\dots,k_d)\in\mathbb{Z}^d,|k_i|\le n}\left|cov(|\triangle_{k,1}X|^{\beta},|\triangle_{\underline{0},1}X|^{\beta})\right|=0, \tag{10}$$

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- There exists a sequence $\{b_n, n\in\mathbb{N}\}$ and a constant C such that $\lim_{n\to+\infty} b_n=0$, $b_{n/2}=O(b_n)$ and

$$\lim_{n\to+\infty}\frac{1}{n^d b_n^2}\sum_{k=(k_1,\dots,k_d)\in\mathbb{Z}^d,|k_i|\le n}\left|cov(|\triangle_{k,1}X|^{\beta},|\triangle_{\underline{0},1}X|^{\beta})\right|\le C^2. \tag{11}$$

Estimation of H and α

Theorem 3

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$$\lim_{n\to+\infty} \widehat{H_n} \stackrel{(\mathbb{P})}{=} H, \lim_{n\to+\infty} \widehat{\alpha_n} \stackrel{(\mathbb{P})}{=} \alpha.$$

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2. *Assume (11), then* $\lim_{n\to+\infty} \widehat{H_n}(\beta) = H, (\mathbb{P}),$

$$\widehat{H_n} - H = O_{\mathbb{P}}(b_n), \widehat{\alpha_n} - \alpha = O_{\mathbb{P}}(b_n).$$

Examples

Theorem 3 is true for:

- Lévy fractional Brownian field with

$$b_n = n^{-d/2}$$

- Well-balanced linear fractional stable field with

$$b_n = \begin{cases} n^{-d/2} & \text{, if } \frac{\alpha H-(L+1)\alpha d}{2} < -d \\ n^{\frac{\alpha H-(L+1)\alpha d}{4}} & \text{, if } -d < \frac{\alpha H-(L+1)\alpha d}{2} < 0 \\ \sqrt{\frac{\ln n}{n}} & \text{, if } \frac{\alpha H-(L+1)\alpha d}{2} = -d \end{cases}$$

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- Takenaka random field with

$$b_n = n^{\frac{\nu-1}{2}}$$

Outline

Definition

Let $0 < \alpha \leq 2$ and $H : U \to (0,1)$ be an infinite differentiable function on a closed interval $U \subset \mathbb{R}$. Let

$$X(t) = \int_{\mathbb{R}} (|t-s|^{H(t)-1/\alpha} - |s|^{H(t)-1/\alpha}) M_\alpha(ds) \qquad (12)$$

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- $0 < \alpha < 2$, $X(t)$ is called *a linear multifractional stable motion*
- $\alpha = 2$, $X(t)$ is called *a multifractional Brownian motion* ($M(du)$ is the standard Gaussian measure on $\mathbb{R}$).

Settings

- Let

$$\triangle_{p,n}X = \sum_{k=0}^{K} a_k X\left(\frac{k+p}{n}\right).$$

- Let γ be fixed such that

$$0 < \limsup_{t\in U} H(t) < \gamma < 1.$$

- Define a set $\nu_{\gamma,n}(u)$ and its cardinal by

$$\nu_{\gamma,n}(u) := \{k \in \mathbb{Z} : \forall p = 0, \dots, K, |\frac{k+p}{n} - u| \leq \frac{1}{n^\gamma}\},$$

$$\upsilon_{\gamma,n}(u) := \#\nu_{\gamma,n}(u)$$

$$\{\frac{k+p}{n}, k \in \nu_{\gamma,n}(u), p = 0, \dots, K\} \subset U.$$

Settings

- Let $\beta \in (-1/2, 0)$ be fixed and

$$V_{u,n}(\beta) = \frac{1}{\upsilon_{\gamma,n}(u)} \sum_{k \in \nu_{\gamma,n}(u)} |\triangle_{k,n} X|^{\beta}$$

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$$\widehat{H}_n(u) := \frac{1}{\beta} \log_2 \frac{V_{u,n/2}(\beta)}{V_{u,n}(\beta)}.$$

Settings

- Let $\beta \in (-1/2, 0)$ be fixed and

$$V_{u,n}(\beta) = \frac{1}{\upsilon_{\gamma,n}(u)} \sum_{k \in \nu_{\gamma,n}(u)} |\triangle_{k,n} X|^{\beta}$$

$$W_{u,n}(\beta) = n^{\beta H(u)} V_{u,n}(\beta).$$

$$\widehat{H}_n(u) := \frac{1}{\beta} \log_2 \frac{V_{u,n/2}(\beta)}{V_{u,n}(\beta)}.$$

- Fix $-1/2 < \beta_1 < \beta_2 < 0$,

$$\widehat{\alpha}_n = \varphi_{-\beta_1,-\beta_2}\left(\psi_{-\beta_1,-\beta_2}(V_{u,n}(\beta_1), V_{u,n}(\beta_2))\right).$$

Estimation of H and α

Theorem 1

Let X be a linear multifractional stable motion or multifractional Brownian motion. For $u \in U$ fixed, then

$$\lim_{n\to+\infty} \widehat{H}_n(u) = H(u), \widehat{H}_n(u) - H(u) = O_{\mathbb{P}}(d_n), \widehat{\alpha}_n - \alpha \quad = O_{\mathbb{P}}(d_n)$$

Estimation of H and α

where d_n is defined by

- $$d_n = \begin{cases} n^{\frac{\alpha(H(u)-\gamma)}{4}} & H(u) < L+1-\frac{2}{\alpha}, H(u) < \gamma \leq \frac{2+\alpha H(u)}{2+\alpha} \\ n^{\frac{\gamma-1}{2}} & H(u) < L+1-\frac{2}{\alpha}, \gamma > \frac{2+\alpha H(u)}{2+\alpha} \\ n^{\frac{\alpha(1-\gamma)(H(u)-(L+1))}{4}} & H(u) > L+1-\frac{2}{\alpha}, \gamma \geq \frac{L+1}{L+2-H(u)} \\ n^{\frac{\alpha(H(u)-\gamma)}{4}} & H(u) > L+1-\frac{2}{\alpha}, H(u) < \gamma < \frac{L+1}{L+2-H(u)} \\ n^{\frac{\alpha(H(u)-\gamma)}{4}} & H(u) = L+1-\frac{2}{\alpha}, H(u) < \gamma < \frac{(L+1)\alpha}{2+\alpha} \\ n^{\frac{\gamma-1}{2}}\sqrt{\ln(n)} & H(u) = L+1-\frac{2}{\alpha}\gamma \geq \frac{(L+1)\alpha}{2+\alpha} \end{cases}$$

  for linear multifractional stable motion

- $$d_n = \begin{cases} n^{H(u)-\gamma} & \text{if } H(u) < \gamma \leq \frac{1+2H(u)}{3} \\ n^{\frac{\gamma-1}{2}} & \text{if } \gamma > \frac{1+2H(u)}{3} \end{cases}$$

  for multifractional Brownian motion.

Perspective

- Improve results in case of multifractional stable motions?
- Other mulfractional multistable processes?
- ...

Thank you for your attention!

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Sketch of proofs - Auxiliary lemmas

(S, μ): a measure space, $f, g \in L^\alpha(S, \mu)$, M: a $S\alpha S$ random measure on S with control measure μ. Set

$$U = \int_S f(s)M(ds), V = \int_S g(s)M(ds), ||U||_\alpha^\alpha = ||V||_\alpha^\alpha = 1,$$

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$$\int_S f(s)g(s)ds \leq \eta < 1, C_\beta = \frac{2^{\beta+1/2}\Gamma(\frac{\beta+1}{2})}{\Gamma(\frac{-\beta}{2})}$$

.

Sketch of proofs - Auxiliary lemmas

Lemma 2

- $$\mathbb{E}|U|^\beta = \frac{C_\beta}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{\mathbb{E}e^{iUy}}{|y|^{1+\beta}} dy,$$
$$\mathbb{E}|U|^\beta |V|^\beta = \frac{C_\beta C_\beta}{2\pi} \int_{\mathbb{R}^2} \frac{\mathbb{E}e^{ixU+iyV}}{|x|^{1+\beta}|y|^{1+\beta}} dxdy$$

in sense of distribution.

Sketch of proofs - Auxiliary lemmas

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  in sense of distribution.
- *There exists a constant $C(\eta)$ such that*
  $$|cov(|U|^{\beta}, |V|^{\beta})| \leq C(\eta) \int_{S} |f(s)g(s)|^{\alpha/2} ds.$$

Sketch of proofs - Auxiliary lemmas

Lemma 3

Let X be a standard $S\alpha S$ random variable with $0 < \alpha \leq 2$, let $-1 < \gamma < 0$ then

$$\mathbb{E}|X|^{\gamma} < +\infty,$$

moreover

$$\mathbb{E}|X|^{\gamma} = \frac{2^{\gamma}\Gamma(\frac{\gamma+1}{2})\Gamma(1-\frac{\gamma}{\alpha})}{\sqrt{\pi}\Gamma(1-\frac{\gamma}{2})}.$$

Sketch of proofs - Auxiliary lemmas

- $g_{u,v}$ is a strictly decreasing function on $(0, +\infty)$ and

$$\lim_{x\to 0} g_{u,v}(x) = 0, \lim_{x\to +\infty} g_{u,v}(x) = -\infty.$$

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- $h_{u,v}$ is invertible and $h_{u,v}^{-1}$ is continuous and differentiable on $(-\infty,0)$.

Sketch of proofs - Auxiliary lemmas

- $\psi_{-\beta_1,-\beta_2}(W_n(\beta_1), W_n(\beta_2))$ converges in probability to $\psi_{-\beta_1,-\beta_2}(\mathbb{E}|\triangle_{0,1}X|^{\beta_1}, \mathbb{E}|\triangle_{0,1}X|^{\beta_2}) = h_{-\beta_1,-\beta_2}(\alpha)$ as $n \to +\infty$.

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- $\psi_{-\beta_1,-\beta_2}(W_n(\beta_1), W_n(\beta_2)) - h_{-\beta_1,-\beta_2}(\alpha) = O_{\mathbb{P}}(b_n)$.
- $\varphi_{-\beta_1,-\beta_2} \circ \psi_{-\beta_1,-\beta_2}$ is continuous at $x_0 = (\mathbb{E}|\triangle_{0,1}X|^{\beta_1}, \mathbb{E}|\triangle_{0,1}X|^{\beta_2})$.