The (β_1, β_2)-skew Brownian motion: an explicit representation of its transition densities and its exact simulation

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Generalized rejection sampling method

Rejection sampling method
Assume to know how to sample the r.v. Y with density $g(x)$.
Then one can sample the r.v. X with density $h(x)$ if:

(i) $\exists M > 0$ such that $h(x) \leq Mg(x)$ for all $x \in \mathbb{R}$;

(ii) $f(x) := \frac{1}{M}\frac{h(x)}{g(x)}$ can be evaluated.

$X \overset{(d)}{=} (Y | U < f(Y))$ i.e. an **exact** simulation is possible.

Theorem

Replacing (ii) by

(ii') *there exists a sequence of functions $(f_n)_n$ converging pointwise to f at a decreasing rate $(\delta_n)_n$.*
Moreover for each $x \in \mathbb{R}$ $f_n(x), \delta_n(x)$ can be evaluated,

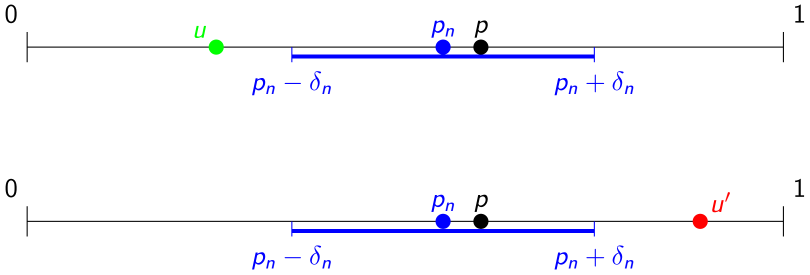
then $X \overset{(d)}{=} (Y | \exists n; U < f_n(Y) - \delta_n(Y))$.

Toy example

Suppose p is an unknown parameter such that

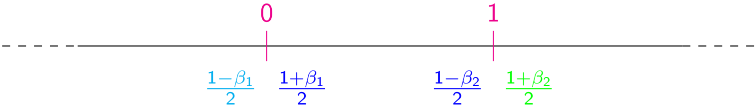
$$\exists (p_n)_n, (\delta_n)_n \ s.t.\ \delta_n \overset{n\to\infty}{\to} 0 \ decreasing,\ and\ |p - p_n| < \delta_n.$$

Then it is possible to simulate exactly a Bernoulli of parameter p since $X := \mathbb{1}_{\{\exists n;\ |U - p_n| > \delta_n,\ U < p_n\}} \sim \mathcal{B}_p$.



Heuristics on the (β_1, β_2)-skew BM

Let $\beta_1, \beta_2 \in [-1, 1]$



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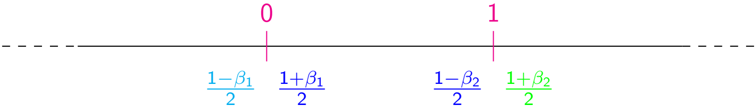
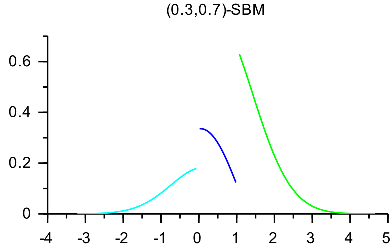
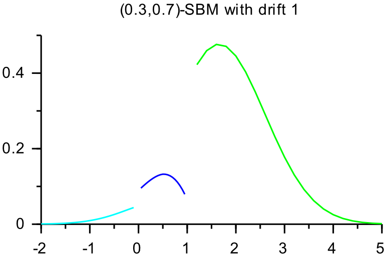


Example: $y \mapsto p^{(0.3,0.7)}(1, 0.5, y)$ is the density of X_1 where $(X_t)_t$ is the $(0.3, 0.7)-$SBM starting at 0.5.





The (β_1, β_2)-SBM with drift

The process is defined for $\beta_1, \beta_2 \in [-1, 1]$, $\mu \in \mathbb{R}$, z_1, $z_2 \in \mathbb{R}$,

- through the SDE, by

$$\begin{cases} dX_t = dW_t + \mu dt + \beta_1 dL_t^{z_1}(X) + \beta_2 dL_t^{z_2}(X); \\ L_t^{z_i}(X) = \int_0^t \mathbb{I}_{\{X_s = z_i\}} dL_s^{z_i}(X) \end{cases}$$

- the infinitesimal generator is $\mathcal{L}f = \frac{1}{2}\Delta f + \mu \nabla f + \sum_{j=1,2} \beta_j \langle \delta_{z_j}, \nabla f \rangle$.

Lemma (Divergence form operator)

$$\begin{cases} \mathcal{L} = \frac{1}{2h(x)} \nabla \left(h(x) \nabla \right) \\ Dom(\mathcal{L}) = \left\{ f \in H_0^1(h(x)dx);\ h\nabla f \in H^1(h^{-1}(x)dx) \right\} \\ h(x) = e^{2\mu x} k(x) \end{cases}$$

$$\text{where } k(x) = \begin{cases} \frac{1}{4}(1-\beta_1)(1-\beta_2) & x < z_1, \\ \frac{1}{4}(1+\beta_1)(1-\beta_2) & z_1 \leq x < z_2, \\ \frac{1}{4}(1+\beta_1)(1+\beta_2) & x \geq z_2. \end{cases}$$

The transition densities of the (β_1, β_2)-SBM

Proposition

$$p^{(\beta_1,\beta_2)}(t,x,y) = p^{(0,0)}(t,x,y) \cdot v^{(\beta_1,\beta_2)}(t,x,y)$$

where, if $z = z_2 - z_1$,

$$v^{(\beta_1,\beta_2)}(t,x,y) = \sum_{k=0}^{\infty} (-\beta_1\beta_2)^k \sum_{j=1}^{4} c_j(y) e^{-\frac{\left(a_j(x,y)+2zk\right)^2}{2t}} e^{-|x-y|\frac{a_j(x,y)+2zk}{t}}$$

$$\begin{cases} c_1(y) \equiv 1 \\ c_2(y) = \left(2\mathbb{1}_{[z_1,+\infty)}(y) - 1\right)\beta_1 \\ c_3(y) = \left(2\mathbb{1}_{[z_2,+\infty)}(y) - 1\right)\beta_2 \\ c_4(y) = \left(1 - 2\mathbb{1}_{[z_1,z_2)}(y)\right)\beta_1\beta_2 \end{cases}$$

$$\begin{cases} a_1(x,y) \equiv 0 \\ a_2(x,y) = |y - z_1| + |x - z_1| - |y - x| \\ a_3(x,y) = |y - z_2| + |y - z_2| - |y - x| \\ a_4(x,y) = 2\left(z_2 - max(x,y,z_1)\right)^+ + 2\left(min(x,y,z_2) - z_1\right)^+ \end{cases}$$

Application of the GRS method

Lemma

There exists an upper bound for $v^{(\beta_1,\beta_2)}(t,x,y)$ uniform in x and y:

$$\sup_{x,y\in\mathbb{R}} \left| v^{(\beta_1,\beta_2)}(t,x,y) \right| \leq \overline{v}_t := \frac{(1+|\beta_1|)(1+|\beta_2|)}{1-|\beta_1\beta_2|e^{-\frac{2z^2}{t}}}$$

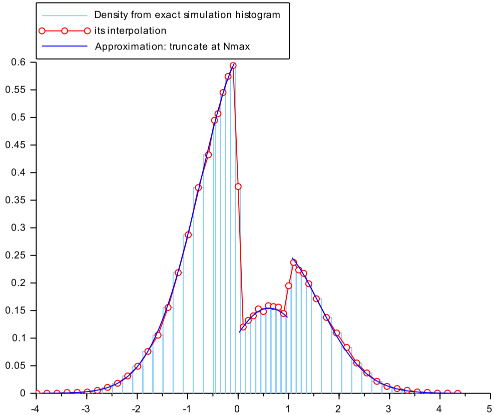
Lemma

The remainder of the truncated series is bounded uniformly in x, y:

$$|R^N v^{(\beta_1,\beta_2)}(t,x,y)| \leq \overline{v}_t \left(|\beta_1\beta_2|e^{-\frac{2z^2}{t}} \right)^{N+1}.$$

Approximation Vs generalized rejection sampling method

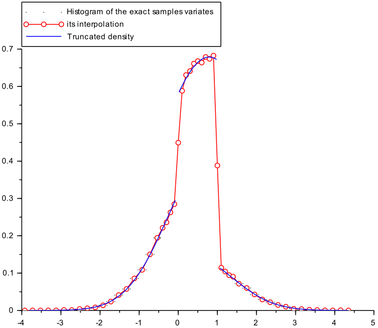
In the simulation we fix $t = 1$ and barriers $z_1 = 0$ and $z_2 = 1$, moreover we will sample 50000 times with GRSM $y \mapsto p^{(-0.7,0.3)}(1, 0.5, y)$



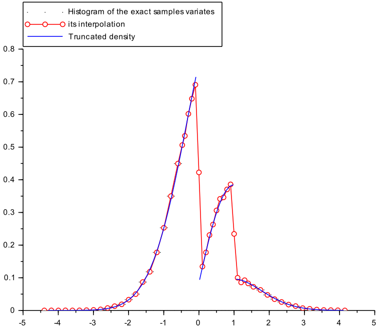
The 50000 simulations through GRS method are EXACT and the density is approximated with its truncation at the 10^{th} term.

Asymmetric cases with $\beta_1\beta_2 > 0$ and $\beta_1\beta_2 < 0$

$y \mapsto p^{(0.3,-0.7)}(1, 0.5, y)$

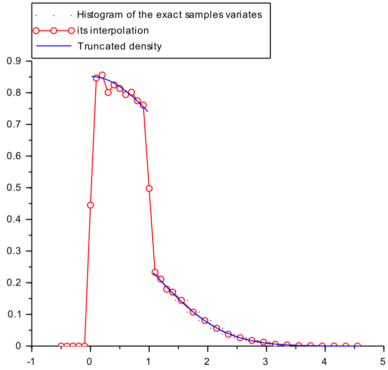


$y \mapsto p^{(-0.8,-0.6)}(1, 0.5, y)$

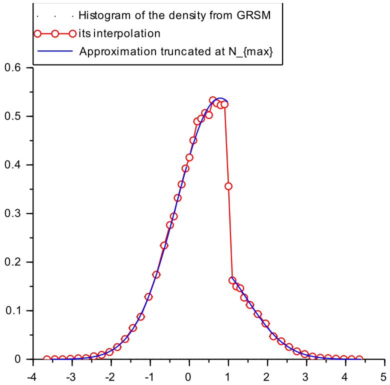


Reflected skew Brownian motion and only one barrier

$y \mapsto p^{(1,-0.5)}(1, 0.5, y)$

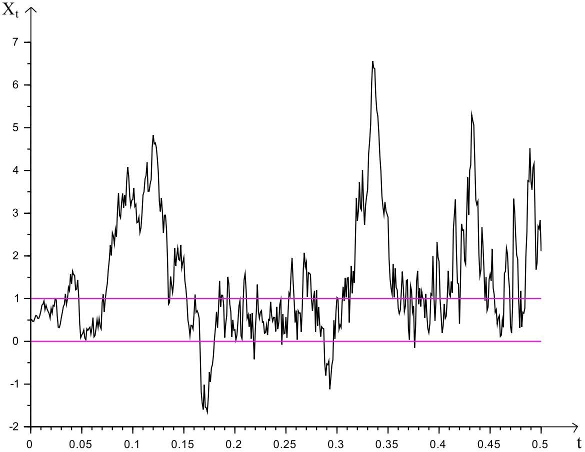


$y \mapsto p^{(0,-0.5)}(1, 0.5, y)$



Simulation of the process $(0.7, -0.2)$-SBM

Assume the initial condition $X_0 = 0.5$



The transition density for the (β_1, β_2)-SBM with drift

Main result:
Assume $\beta_1, \beta_2 \in (-1, 1)$ and $\mu \in \mathbb{R}$.

$$p_\mu^{(\beta_1,\beta_2)}(t,x,y) = p_\mu^{(0,0)}(t,x,y) v_\mu^{(\beta_1,\beta_2)}(t,x,y)$$

where the function $v_\mu^{(\beta_1,\beta_2)}$ is given by a series of Fourier transforms. It is a series involving two functions

$$\begin{cases} J_0(\omega, \beta_i \mu \sqrt{t}) := e^{-\frac{\omega^2}{2}} \sqrt{2\pi} e^{\frac{(\beta_i \mu \sqrt{t}+\omega)^2}{2}} \Phi^c(\omega + \beta_i \mu \sqrt{t}) \text{ with } i \in \{1,2\} \\ J_1(\omega) := -e^{-\frac{\omega^2}{2}}, \end{cases}$$

evaluated in $\omega \in \left\{ \omega_{j,k} := \frac{a_j(x,y)+2zk+|y-x|}{\sqrt{t}},\ j = 1,2,3,4,\ k \in \mathbb{N} \right\}$.

Representation of the transition density

Proposition

The infinitesimal generator $\mathcal{L}$ is self adjoint in $L^2(h(x)dx)$, and the following equality holds:

$$\boxed{p_{\mu}^{(\beta_1,\beta_2)}(t,x,y) = \frac{1}{2\pi i}\int_{\Gamma} e^{\lambda t} G_{\mu}^{(\beta_1,\beta_2)}(x,y;\lambda)d\lambda,}$$

where Γ is a complex contour of $\sigma(\mathcal{L}) \subseteq (-\infty, 0]$ and for $y \in \mathbb{R}$ and $\lambda \in \mathbb{C} \setminus (-\infty, 0]$, $x \mapsto G_{\mu}^{(\beta_1,\beta_2)}(x,y;\lambda)$, the Green's function for the resolvent, solves

$$\begin{cases} (\lambda - \mathcal{L})u(x) = \delta_{\{y\}}(x), & u \in \mathcal{C}^2(\mathbb{R} \setminus \{z_1, z_2\}) \cap \mathcal{C}(\mathbb{R}) \\ h(z_i^+)u'(z_i^+) = h(z_i^-)u'(z_i^-), & i = 1, 2. \end{cases}$$

This is known as the Titchmarsh-Kodaira-Yoshida method ($\mu = 0$).

The Green function for the (β_1, β_2)-SBM with drift

Lemma

The Green functions, if $w := \sqrt{2\lambda + \mu^2} \in \{v \in \mathbb{C}; \Re(v) > 0\}$, *are*

$$G(x, y; w) = \frac{1}{w} e^{\mu(y-x)} \frac{\sum_{j=1}^{4} c_j(\mu, y; w) e^{-w(a_j(x,y)+|x-y|)}}{\beta_1\beta_2 e^{-2wz}(w^2 - \mu^2) + (w + \beta_1\mu)(w + \beta_2\mu)}.$$

$c_j(\mu, y; w) = w^2 c_{j,0}(y) + w\mu c_{j,1}(y) + \mu^2 c_{j,2}(y)$, *where*

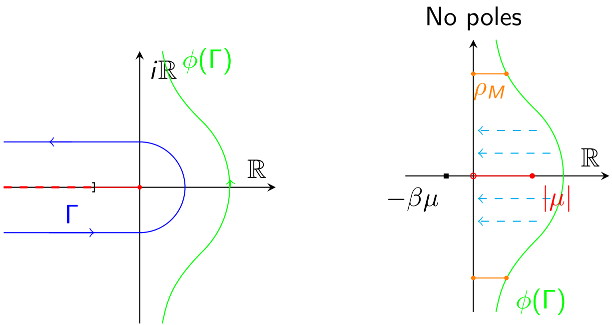
$$\begin{cases} c_{1,0} = 1, \\ c_{2,0} = \left(2\mathbb{1}_{[z_1,+\infty)}(y) - 1\right)\beta_1 \\ c_{3,0} = \left(2\mathbb{1}_{[z_2,+\infty)}(y) - 1\right)\beta_2 \\ c_{4,0} = \left(1 - 2\mathbb{1}_{[z_1,z_2)}(y)\right)\beta_1\beta_2 \end{cases} \quad \begin{cases} c_{1,1} = \beta_1 + \beta_2 \\ c_{2,1} = -\beta_1 - c_{4,0} \\ c_{3,1} = -\beta_2 + c_{4,0} \\ c_{4,1} = 0 \end{cases} \quad \begin{cases} c_{1,2} = \beta_1\beta_2 \\ c_{2,2} = \beta_1 c_{3,0} \\ c_{3,2} = -\beta_2 c_{2,0} \\ c_{4,2} = -c_{4,0}. \end{cases}$$

Ideas of the proof

If $\phi(\lambda) = \sqrt{2\lambda + \mu^2}$, we recall the contour integral representation

$$p_\mu^{(\beta_1,\beta_2)}(t,x,y) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t} G_\mu^{(\beta_1,\beta_2)}(x,y;\phi(\lambda))d\lambda,$$

and we will use techniques of complex analysis:



Towards a series of Fourier transforms

For simplicity $\beta_1\mu > 0, \beta_2\mu > 0$, hence there is no pole.
The formula for ${v_\mu}^{(\beta_1,\beta_2)}(t,x,y)$ is

$$\frac{\sqrt{t}}{\sqrt{2\pi}} e^{\frac{(x-y)^2}{2t}} \int_{\mathbb{R}} e^{-\frac{u^2}{2}t} \frac{\sum_{j=1}^{4} -c_j(\mu, y; iu) e^{-iu(a_j(x,y)+|x-y|)}}{\beta_1\beta_2 e^{-i2zu}(u^2+\mu^2) + (u - i\beta_1\mu)(u - i\beta_2\mu)} du.$$

Remark

For all $u, \mu \in \mathbb{R}$ and $\beta_1, \beta_2 \in (-1,1)$

$$|\beta_1\beta_2| (u^2 + \mu^2) \leq |(u - i\beta_1\mu)(u - i\beta_2\mu)|$$

Therefore

$$\frac{1}{1 + \frac{\beta_1\beta_2(u^2+\mu^2)e^{-i2zu}}{(u-i\beta_1\mu)(u-i\beta_2\mu)}} = \sum_{k=0}^{\infty} \left(\frac{-\beta_1\beta_2(u^2+\mu^2)}{(u-i\beta_1\mu)(u-i\beta_2\mu)} \right)^k e^{-i2zuk}$$

Density's uniform bound

Under the assumption $\beta_1\mu > 0, \ \beta_2\mu > 0$,

$$\begin{cases} v_\mu^{(\beta_1,\beta_2)}(t,x,y) = e^{\frac{|x-y|^2}{2t}} \sum_{k=0}^{\infty} \sum_{j=1}^{4} F_{j,k}(\omega_{j,k}), \quad \omega_{j,k} := \frac{a_j(x,y)+2zk+|y-x|}{\sqrt{t}} \\ F_{j,k} := (-\beta_1\beta_2)^k \ \mathcal{F}\left(w \mapsto e^{-\frac{w^2}{2}} c_j(y,\mu\sqrt{t}; iw) \cdot \frac{-(w^2+\mu^2 t)^k}{(w-i\beta_1\mu\sqrt{t})^{k+1}(w-i\beta_2\mu\sqrt{t})^{k+1}}\right). \end{cases}$$

Lemma

There exists an upper bound for $v_\mu^{(\beta_1,\beta_2)}(t,x,y)$ uniform in x and y:

$$\sup_{x,y\in\mathbb{R}} \left| v_\mu^{(\beta_1,\beta_2)}(t,x,y) \right| \le \frac{C(\beta_1,\beta_2)}{1-e^{-\frac{2z^2}{t}}}$$

$$|R^N v_\mu^{(\beta_1,\beta_2)}(t,x,y)| \le \frac{C(\beta_1,\beta_2)}{1-e^{-\frac{2z^2}{t}}} \, e^{-\frac{2z^2}{t}(N+1)}.$$

Remerciements

Merci pour votre attention!

Bibliography

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