# A risk management approach to capital allocation

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# Outline

A risk management approach to capital allocation


Multivariate risk theory :

- Dependence modeling ;
- Multivariate ruin probabilities ;
- Multivariate risk measures...

What is a capital allocation ?

- Euler and Shapley principles ([Tasche, 2007],[Denault, 2001]).
- Minimization of some ruin probabilities or multivariate risk indicators.

# What is a capital allocation ?

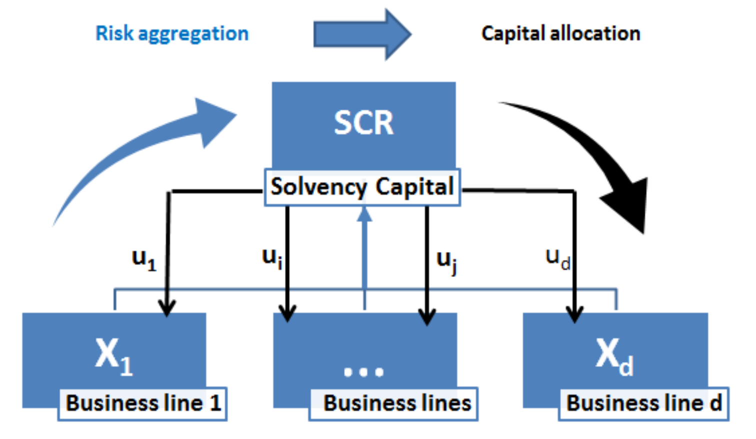


FIGURE: What is a capital allocation ?

## 2 Optimal allocation

- Multivariate risk indicators
- The allocation method
- Optimality conditions
- Penalty functions

# Multivariate risk framework

- We consider a vectorial risk process $X^p = (X_1^p, \ldots, X_d^p)$, where $X_k^p$ corresponds to the losses of the $k^{th}$ business line during the $p^{th}$ period.
- We denote by $R_k^p$ the reserve of the $k^{th}$ line at time $p$, so :

  $R_k^p = u_k - \sum_{l=1}^{p} X_k^l$, where $u_k \in \mathbb{R}^+$ is the initial capital of the $k^{th}$ business line ;
- $u = u_1 + \cdots + u_d$ is the initial capital of the group ;
- $d$ is the number of business lines.
- $\mathcal{U}_u^d = \{v = (v_1, \ldots, v_d) \in [0, u]^d, \sum_{i=1}^d v_i = u\}$ is the set of possible allocations of the initial capital $u$.
- $\forall i \in \{1, \ldots, d\}$ let $\alpha_i = \frac{u_i}{u}$, then, $\sum_{i=1}^d \alpha_i = 1$ if $(u_1, \ldots, u_d) \in \mathcal{U}_u^d$.
- $X_k$ corresponds to the losses of the $k^{th}$ branch during one period $(n = 1)$.

Cénac et al. (2012) defined the two following multivariate risk indicators, for $d$ risks and $n$ periods, given penalty functions $g_k,\ k \in \{1,\dots,d\}$ :

- the indicator $I$ :

$$I(u_1,\dots,u_d) = \sum_{k=1}^{d} \mathbb{E}\left(\sum_{p=1}^{n} g_k(R_k^p)\, \mathbb{1}_{\{R_k^p<0\}}\, \mathbb{1}_{\{\sum_{j=1}^{d} R_j^p>0\}}\right),$$

- the indicator $J$ :

$$J(u_1,\dots,u_d) = \sum_{k=1}^{d} \mathbb{E}\left(\sum_{p=1}^{n} g_k(R_k^p)\, \mathbb{1}_{\{R_k^p<0\}}\, \mathbb{1}_{\{\sum_{j=1}^{d} R_j^p<0\}}\right),$$

$g_k : \mathbb{R}^- \to \mathbb{R}^+$ are $C^1$, convex functions with $g_k(0) = 0,\ g_k(x) \geq 0$, $k = 1,\dots,d$, $g_k$ are decreasing functions on $\mathbb{R}^-$.

# Optimal allocation
## Multivariate risk indicators

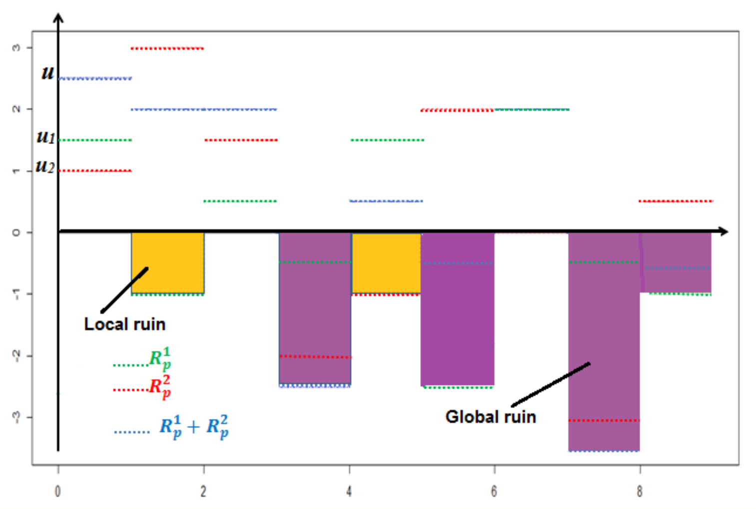


FIGURE: Multivariate risk indicators

# The allocation method

Since the new regulation, such as Solvency 2, require a one year allocation strategy, in this paper we focus on a single period ($n = 1$).

**Definition : Optimal allocation**

Let $\mathbf{X}$ be a non negative random vector of $\mathbb{R}^d$, $u \in \mathbb{R}^+$ and $\mathcal{K}_{\mathbf{X}} : \mathcal{U}_u^d \to \mathbb{R}^+$ a multivariate risk indicator associated to $\mathbf{X}$ and $u$. An optimal allocation of the capital $u$ for the risk vector $\mathbf{X}$ is defined by :

$$(u_1, \ldots, u_d) \in \underset{(v_1,\ldots,v_d)\in\mathcal{U}_u^d}{\arg\inf} \left\{\mathcal{K}_{\mathbf{X}}(v_1, \ldots, v_d)\right\}.$$

- For risk indicators of the form $\mathcal{K}_{\mathbf{X}}(v) = \mathbb{E}[S(\mathbf{X}, \mathbf{v})]$, with a scoring function $S : {\mathbb{R}^+}^d \times {\mathbb{R}^+}^d \to \mathbb{R}^+$, this definition can be seen as an extension in a multivariate framework of the concept of elicitability.
- For an initial capital $u$, and an optimal allocation minimizing the multivariate risk indicator $I$, we seek $u^* \in \mathbb{R}_+^d$ such that :

$$I(u^*) = \inf_{v_1+\cdots+v_d=u} I(v), \quad v \in \mathbb{R}_+^d.$$

# Assumptions

## Assumptions

- **H1** $\mathcal{K}_{\mathbf{X}}$ admits a unique minimum in $\mathcal{U}_u^d$. In this case, we denote by $A_{X_1,\ldots,X_d}(u) = (u_1,\ldots,u_d)$ the optimal allocation of $u$ on the $d$ risky branches in $\mathcal{U}_u^d$.
- **H2** The functions $g_k$ are differentiable and such that for all $k \in \{1,\ldots,d\}$, $g_k'(u_k - X_k)$ admits a moment of order one, and $(X_k, S)$ has a joint density distribution denoted by $f_{(X_k,S)}$.
- **H3** The $d$ risks have the same penalty function $g_k = g, \forall k \in \{1,\ldots,d\}$.

- The first assumption is verified when the indicator is strictly convex, this is particularly true if at least one function $g_k$ is strictly convex ; and the joint density $f_{(X_k,S)}$ support contains $[0,u]^2$.

# Optimality condition

- Under assumption H2, the risk indicators $I$ and $J$ are differentiable,

$$(\nabla I(v))_i = \sum_{k=1}^{d} \int_{v_k}^{+\infty} g_k(v_k - x) f_{X_k,S}(x,u)dx + \mathbb{E}[g_i'(v_i - X_i) \mathbb{1}_{\{X_i > v_i\}} \mathbb{1}_{\{S \leq u\}}]$$

and,

$$(\nabla J(v))_i = \sum_{k=1}^{d} \int_{v_k}^{+\infty} g_k(v_k - x) f_{X_k,S}(x,u)dx + \mathbb{E}[g_i'(v_i - X_i) \mathbb{1}_{\{X_i > v_i\}} \mathbb{1}_{\{S \geq u\}}].$$

- Under H1 and H2, using Lagrange multipliers, we obtain an optimality condition verified by the unique solution,

$$\mathbb{E}[g_i'(u_i - X_i) \mathbb{1}_{\{X_i > u_i\}} \mathbb{1}_{\{S \leq u\}}] = \mathbb{E}[g_j'(u_j - X_j) \mathbb{1}_{\{X_j > u_j\}} \mathbb{1}_{\{S \leq u\}}], \ \forall (i,j) \in \{1, \ldots, d\}^2$$

- A natural choice for penalty functions is the ruin severity : $g_k(x) = |x|$.
- If the joint density $f_{(X_k,S)}$ support contains $[0,u]^2$, for at least one $k \in \{1,\ldots,d\}$, our optimization problem has a unique solution.
- We may write the indicators as follows :

$$I(u_1,\ldots,u_d) = \sum_{k=1}^{d} \mathbb{E}\left((X_k - u_k)_+ \mathbb{1}_{\{S\leq u\}}\right),$$

and,

$$J(u_1,\ldots,u_d) = \sum_{k=1}^{d} \mathbb{E}\left((X_k - u_k)_+ \mathbb{1}_{\{S\geq u\}}\right).$$

- The optimality condition :

$$\mathbb{P}\left(X_i > u_i, S \leq u\right) = \mathbb{P}\left(X_j > u_j, S \leq u\right), \forall (i,j) \in \{1,2,\ldots,d\}^2.$$

For the $J$ indicator, this condition can be written :

$$\mathbb{P}\left(X_i > u_i, S \geq u\right) = \mathbb{P}\left(X_j > u_j, S \geq u\right), \forall (i,j) \in \{1,2,\ldots,d\}^2.$$

## 3 Coherence properties

- Coherence
- Other desirable properties
- Coherence of the optimal allocation

# Coherence

- Following Artzner et al. (1999) [Artzner et al., 1999] and Denault (2001)[Denault, 2001] we reformulate coherence axioms in a more general multivariate context.

## Definition : Coherent allocation

A capital allocation $(u_1, \dots, u_d) = A_{X_1,\dots,X_d}(u)$ of an initial capital $u \in \mathbb{R}^+$ is coherent if it satisfies the following properties :

1. **Full allocation :**

$$\sum_{i=1}^{d} u_i = u.$$

2. **Riskless allocation :** For a deterministic risk $X = c$, where the constant $c \in \mathbb{R}^+$ :

$$A_{X,X_1,\dots,X_d}(u) = (c, A_{X_1,\dots,X_d}(u - c)).$$

## Definition : Coherent allocation

3. **Symmetry :** if

$$(X_1, \dots, X_{i-1}, X_i, X_{i+1}, \dots, X_{j-1}, X_j, X_{j+1}, \dots, X_d)$$
$$\stackrel{\mathcal{L}}{=}$$
$$(X_1, \dots, X_{i-1}, X_j, X_{i+1}, \dots, X_{j-1}, X_i, X_{j+1}, \dots, X_d),$$

   then $u_i = u_j$.

4. **Sub-additivity :** $\forall M \subseteq \{1, \dots, d\}$, let $(u^*, u_1^*, \dots, u_r^*) = A_{\sum_{i \in M} X_i, X_{j \in \{1,\dots,d\} \setminus M}}(u)$, where $r = d - card(M)$ and $(u_1, \dots, u_d) = A_{X_1,\dots,X_d}(u)$ :

$$u^* \leq \sum_{i \in M} u_i.$$

5. **Comonotonic additivity :** For $r \leqslant d$ comonotonic risks,

$$A_{X_{i_{i \in \{1,\dots,d\} \setminus CR}}, \sum_{k \in CR} X_k}(u) = (u_{i_{i \in \{1,\dots,d\} \setminus CR}}, \sum_{k \in CR} u_k),$$

   where $CR$ denotes the set of the $r$ comonotonic risk indexes.

# Desirable properties

## Positive homogeneity

An allocation is positively homogeneous, if for any $\alpha \in \mathbb{R}^+$, it satisfies :

$$A_{\alpha X_1,\ldots,\alpha X_d}(\alpha u) = \alpha A_{X_1,\ldots,X_d}(u).$$

## Translation invariance

An allocation is invariant by translation, if for all $(a_1,\ldots,a_d) \in \mathbb{R}^d$ such that $u > \sum_{k=1}^d a_k$, it satisfies :

$$A_{X_1+a_1,\ldots,X_d+a_d}(u) = A_{X_1,\ldots,X_d}\left(u - \sum_{k=1}^d a_k\right) + (a_1,\ldots,a_d).$$

## Continuity

An allocation is continuous, if for all $i \in \{1,\ldots,d\}$ :

$$\lim_{\epsilon \to 0} A_{X_1,\ldots,(1+\epsilon)X_i,\ldots,X_d}(u) = A_{X_1,\ldots,X_i,\ldots,X_d}(u).$$

# Desirable properties

- We recall the definition of the order stochastic dominance, as it is presented in Shaked and Shanthikumar (2007)[Shaked and Shanthikumar, 2007].
- For random variables $X$ and $Y$, $Y$ first-order stochastically dominates $X$ if and only if :

$$\bar{F}_X(x) \leq \bar{F}_Y(x), \quad \forall x \in \mathbb{R}^+,$$

  and in this case we denote : $X \leq_{st} Y$.
- This definition is also equivalent to the following one :

$$X \leq_{st} Y \Leftrightarrow \mathbb{E}[u(X)] \leq \mathbb{E}[u(Y)], \text{for all } u \text{ increasing function.}$$

## Monotonicity

An allocation satisfies the monotonicity property, if for $(i,j) \in \{1, \ldots, d\}^2$ :

$$X_i \leq_{st} X_j \Rightarrow u_i \leq u_j.$$

# Main theoretical result

## Coherence of the optimal allocation

- In the case of penalty functions $g_k(x) = |x| \; \forall k \in \{1, \ldots, d\}$, and for continuous random vector $(X_1, \ldots, X_d)$, such that the joint density $f_{(X_k,S)}$ support contains $[0,u]^2$, for at least one $k \in \{1, \ldots, d\}$, the optimal allocation by minimization of the indicators $I$ and $J$ is a **symmetric riskless full allocation**. It satisfies the properties of **comonotonic additivity**, **positive homogeneity**, **translation invariance**, **monotonicity**, and **continuity**.

- General results are presented in Maume-Deschamps, Rullière and S (2016) [Maume-Deschamps et al., 2016b].
- The optimal allocation method may be used for the economic capital allocation between the different branches of a group.

# What could be the best choice for a capital allocation ?

- The optimal allocation can be considered coherent from an economic point of view ;
- The first goal of the Solvency 2 norms is the insurers' protection ;
- The classical methods of risk allocation give weight to each business line in the group risk.
- The optimal allocation is based on a global risk optimization.
- The capital allocation by minimizing a risk indicator seems more in line the with Solvency 2 goals.
- Conventional capital allocation methods are based on a chosen univariate risk measure and their properties are derived from those of this risk measure.
- It seems more coherent in a multivariate framework to use directly a multivariate risk indicator, not only for risk measurement, but also for capital allocation.
- Another important criterion in the choice of the allocation method is the nature of the capital.
- The best allocation method choice depends finally on the risk aversion of the insurer.

# Conclusion

- In this article, we have shown that the capital allocation method by minimization of some multivariate risk indicators can be considered as coherent from an economic point of view ;
- In the case of the proposed optimal capital allocation, the risk management is at the heart of the allocation process. That is why we think that from a risk management point of view, this method can be considered as more flexible.
- Allocation by minimizing the indicators $I$ and $J$ is studied in higher dimension in Maume-Deshamps et al. (2016) ;
- Its behavior and asymptotic behavior for some special distributions' families are also analyzed in [Maume-Deschamps et al., 2016a] ;
- The impact of dependence on the allocation composition is studied in the same paper.
- Finally, the choice of a capital allocation method remains a complex and crucial exercise.

# Thank you for your attention!

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