Toroidal dimer model and Temperley's bijection

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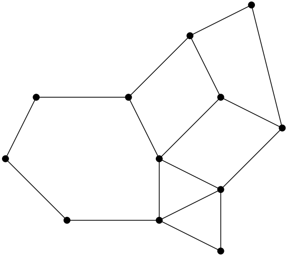
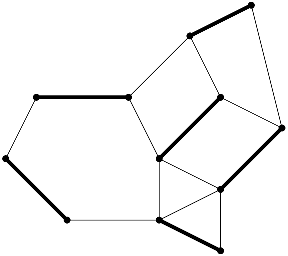
Plan

- Graph : $G = (V(G), E(G))$



- Graph : $G = (V(G), E(G))$
- Dimer configuration

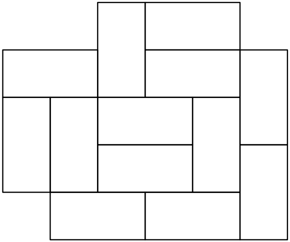


- Edges of G are assigned a positive weight function $c(e)$.
- The dimer Boltzmann measure of a dimer configuration M is :

$$\mathbb{P}(M) = \frac{\prod_{e \in M} c(e)}{Z}$$

$$Z = \sum_{M \in \mathcal{M}(G)} \prod_{e \in M} c(e).$$

- Dimer model and domino tiling



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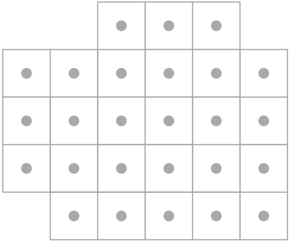

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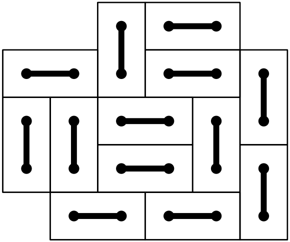
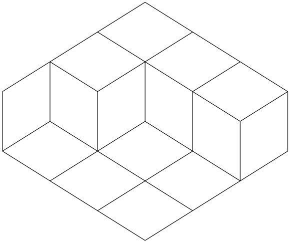


- Dimer model and lozenge tiling



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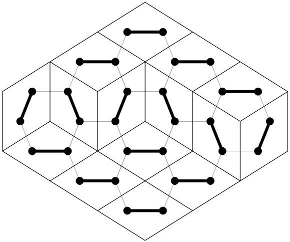
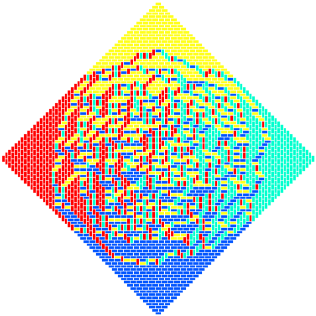
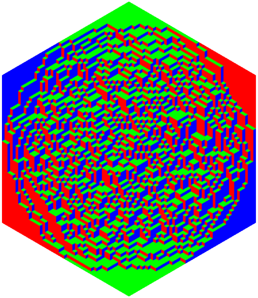


The limit shape of dimer configurations for bounded planar graph.

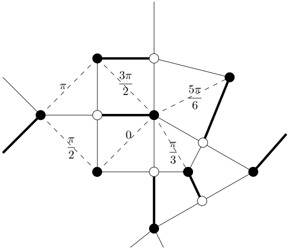


By Cohn



By Linde, Moore and Nordahl

- Height function of a dimer configuration on bipartite graph.
- On a Temperley graph, one definition of the height function is by turning angle.



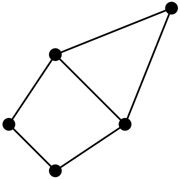
- On torus, this induces a height change (h_x^M, h_y^M).

Kenyon, Okounkov, Sheffield (2003) :

- On a toroidal bipartite graph, choose a path γ_x (resp. γ_y) on the dual of the graph winding once horizontally (resp. vertically).
- By adding a magnetic field $B = (B_x, B_y)$, we mean multiplying the edges crossing γ_x by e^{B_x} if the black vertex is on the left of γ_x and by e^{-B_x} if on the right of γ_x. Same for edges crossing γ_y.
- The modification of the weight of a configuration caused by B only depends on its height change.
- Let G be a periodic bi-partite graph and $G_n = G/(n\mathbb{Z})^2$. Dimer measures on G_n converge to an ergodic Gibbs measure μ with slope (s, t).
- By varying B, we get measures of all possible slopes.

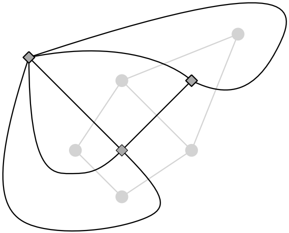
Plan

- Primal graph, dual graph and double graph



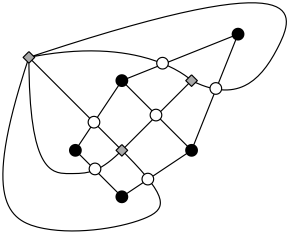
Primal graph G

- Primal graph, dual graph and double graph



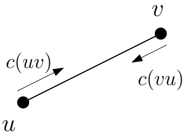
Dual graph G^*

- Primal graph, dual graph and double graph

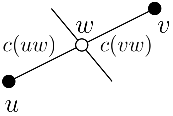


Double graph G^d

Weight setting of a double graph G^d (Temperley Graph)



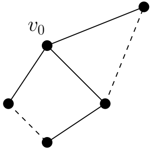
Weights of G



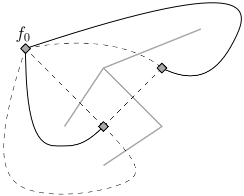
Weights of G^d

In the figure above, we set $c(uv) = c(uw)$, $c(vu) = c(vw)$, and edges of G^* are of weight 1.

- Temperley's bijection on planar graph (Temperley 1974, Kenyon, Propp, Wilson 2000)

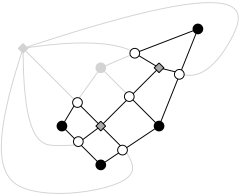


Primal tree T

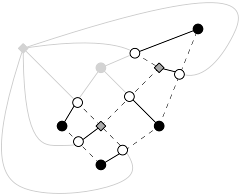


Dual tree T^*

- Temperley's bijection on planar graph (Temperley 1974, Kenyon, Propp, Wilson 2000)

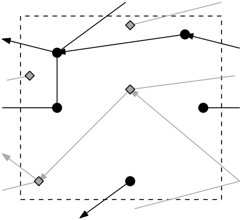


Graph $G^d(v_0, f_0)$

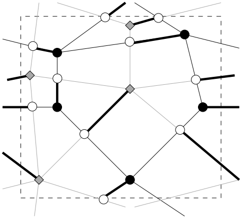


Dimer configuration

- Temperley's bijection on toroidal graph : oriented cycle rooted spanning forest (*CRSF*)



Oriented *CRSF*



Dimer configuration on G^d

- The measure of oriented spanning tree pair T, T^* (resp. oriented *CRSF* pair (F, F^*)) :

$$\mathbb{P}(T, T^*) = \frac{\prod_{\vec{e} \in T} c(\vec{e}) \prod_{\vec{e^*} \in T^*} c(\vec{e^*})}{Z_{\mathcal{T}(\mathcal{G}, \mathcal{G}^*)}}$$

$$\mathbb{P}(F, F^*) = \frac{\prod_{\vec{e} \in F} c(\vec{e}) \prod_{\vec{e^*} \in F^*} c(\vec{e^*})}{Z_{\mathcal{F}(\mathcal{G}, \mathcal{G}^*)}}$$

- By summing over all possible duals, the second one gives a measure of oriented *CRSF* of *G*.

Plan

Theorem (Pemantle 1991)

The uniform spanning tree measures on $\mathbb{Z}^d \bigcap [-n, n]^2$ *converge weakly as* $n \to \infty$. *When* $d \leq 4$, *the limiting measure is supported a.s. by spanning trees. When* $d \geq 5$ *the spanning forest has a.s. infinitely connected components.*

- Result generalized by Benjamini, Lyons, Peres and Schramm for non-oriented planar graphs.

How about the oriented *CRSF* measure ? Consider a $\mathbb{Z}^2$-periodic oriented graph G, and let $G_n = G/(n\mathbb{Z})^2$, $G_n^d = G^d/(n\mathbb{Z})^2$.

- Convergence of the oriented *CRSF* measure on G_n is clear by the corresponding results in the dimer model.
- Height function of a dimer configuration $\leftrightarrow$ winding along a path + jumps + reversions. So the height change of a configuration is equal to the signed sum of the homology class of corresponding oriented *CRSF* pair.

- The *Laplacian* associated to a connection Φ is the operator $\Delta^{\Phi} : \mathbb{C}^V \to \mathbb{C}^V$ defined by

$$\Delta^{\Phi} f(v) = \sum_{u \sim v} c_{vu}(f(v) - \phi_{uv} f(u)).$$

- A decomposition $\Delta^{\Phi} = d^* d$, where

$$df(\vec{e}) = \phi_{ve} f(v) - \phi_{v'e} f(v'),$$

$$d^*(\omega)(v) = \sum_{\vec{e}=v'v} c_{vv'} \phi_{ev} \omega(\vec{e}).$$

- Magnetic field B on $G^d \leftrightarrow$ connection Φ on G.

Define $\mathcal{C}_0(\mathbb{Z}^2)$ as the space of G_1^d-vector-valued functions decaying at infinity, and define $\mathcal{C}_0^B(\mathbb{Z}^2)$ as its (magnetic field B) modified version :

$$\mathcal{C}_0^B(\mathbb{Z}^2) := \{f : \mathbb{Z}^2 \to \mathcal{G}_1^d : e^{xB_y+yB_x} f(x,y;v) \in \mathcal{C}_0(\mathbb{Z}^2)\}.$$

Theorem

The measure μ is a determinantal process, whose kernel is the unique infinite matrix A such that every row $A_e \in \mathcal{C}_0^B(\mathbb{Z}^2)$ and $A\Delta^\Phi = d$.

When the slope is non-zero,

Theorem

When the slope of the limiting dimer measure is non-zero, then under μ, there are a.s. infinitely many connected components.

There are infinite bands. Conditioned on the boundaries of bands, the interiors are weighted spanning forests.

When the slope is zero ?

Lemma

In the phase diagram of the dimer measure of G^d, the point $B = (0,0)$ always corresponds to a zero slope.

- Same slope $\leftrightarrow$ same measure.
- When $B = (0,0)$, we can approach the *CRSF* measures by spanning tree measures on planar graph, and Wilson's algorithm characterize the properties of spanning trees.

Let $G_n = G/(n\mathbb{Z}^2)$ and $\overline{G}_n = G\bigcap[-n,n]^2$.

- Consider the wired spanning tree measure on $\overline{G}_n$.
- If it converges when $n \to \infty$ and its kernel decays at infinity, then by the theorem of the uniqueness, this is the same measure as μ. This is equivalent to the convergence and decay of

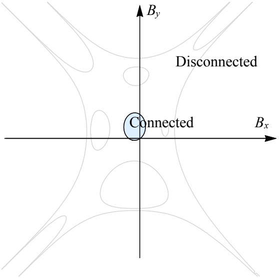
$$(A_N)_{w,v} = \mathbb{E}\left[\#RW_{v_2}^{G_N} \text{ visits } v - \#RW_{v_1}^{G_N} \text{ visits } v\right]$$

- The condition above is denoted by $(\star)$. It is verified by graphs transient, graphs non-oriented, ect.

Theorem

Let G be a graph verifying the condition $(\star)$. When the slope of the limiting dimer measure is zero, then the CRSF measures on G_n and wired spanning tree measure on $\overline{G}_n$ converge to the same measure μ. Under μ, there is a.s. one connected spanning tree.

Phase diagram



Thank you for your attention