Uniqueness and propagation of chaos for the Boltzmann equation with moderately soft potentials

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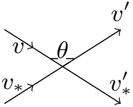
The Boltzmann equation (3D-spatially homogeneous):

$$\partial_t f_t(v) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(|v - v_*|, \theta) \, [f_t(v') f_t(v'_*) - f_t(v) f_t(v_*)] dv_* d\sigma,$$

with $v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma, \; v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma.$



- $f_t(v)$ - the density of particles with velocity v at t in a dilute gas
- v, v_*-pre-collisional velocities
- v', v'_*-post-collisional velocities
- θ-deviation angle defined by $\cos\theta = \frac{v-v_*}{|v-v_*|} \cdot \sigma$
- $B(|v - v_*|, \theta)$-the collision kernel

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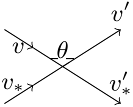

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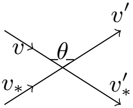

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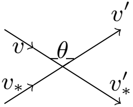

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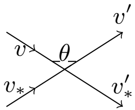

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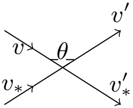

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Assumption for collision kernel B:

For some $\nu \in (0,1)$, and $\gamma \in (-1,0)$ with $\gamma + \nu > 0$, assume that there is a measurable function $\beta : (0,\pi] \to \mathbb{R}_+$ such that

$$B(|v - v_*|, \theta) \sin\theta = |v - v_*|^\gamma \beta(\theta),$$
$$\forall\ \theta \in (0, \pi/2], \beta(\theta) \sim \theta^{-1-\nu},$$
$$\forall\ \theta \in (\pi/2, \pi),\ \ \beta(\theta) = 0.$$

- A classical physical example: inverse power laws interactions, i.e. a repulsive force $\sim 1/r^s$ for some $s > 2$.
- It then yields $\gamma = (s-5)/(s-1)$ and $\nu = 2/(s-1)$.
- Hard potentials $(s > 5, \gamma > 0)$, Maxwell molecules $(s = 5, \gamma = 0)$, soft potentials $(2 < s < 5, -3 < \gamma < 0)$.

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Parameterization (Fournier-Méléard '02):

- For each $x \in \mathbb{R}^3 \setminus \{0\}$, consider an orthonormal basis of $\mathbb{R}^3$ $(\frac{x}{|x|}, \frac{I(x)}{|x|}, \frac{J(x)}{|x|})$ with $|I(x)| = |J(x)| = |x|$.
- For $x, v, v_* \in \mathbb{R}^3$, $\theta \in (0, \pi]$, $\varphi \in [0, 2\pi)$, set

$$\begin{cases} \Gamma(x,\varphi) := (\cos\varphi) I(x) + (\sin\varphi) J(x), \\ v'(v, v_*, \theta, \varphi) := v - \frac{1-\cos\theta}{2}(v - v_*) + \frac{\sin\theta}{2}\Gamma(v - v_*, \varphi), \\ a(v, v_*, \theta, \varphi) := v'(v, v_*, \theta, \varphi) - v. \end{cases}$$

- $\Gamma(x, \varphi)$ is orthogonal to x and has the same norm as x.
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$\sigma = \frac{v-v_*}{|v-v_*|}\cos\theta + \frac{I(v-v_*)}{|v-v_*|}\sin\theta\cos\varphi + \frac{J(v-v_*)}{|v-v_*|}\sin\theta\sin\varphi.$

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The weak solution to the Boltzmann equation:

If a measurable family of probability measures $(f_t)_{t\geq 0}$ on $\mathbb{R}^3$ satisfies the following two conditions:

- For all $t \geq 0$,

$$\int_{\mathbb{R}^3} v f_t(dv) = \int_{\mathbb{R}^3} v f_0(dv) \text{ and } \int_{\mathbb{R}^3} |v|^2 f_t(dv) = \int_{\mathbb{R}^3} |v|^2 f_0(dv) < \infty.$$

- For any bounded globally Lipschitz function $\phi : \mathbb{R}^3 \mapsto \mathbb{R}$, any $t \in [0, T]$,

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Some notation:

- $\mathcal{P}(\mathbb{R}^3)$: the set of probability measures on $\mathbb{R}^3$.
- For $q > 0$, $\mathcal{P}_q(\mathbb{R}^3) = \{f \in \mathcal{P}(\mathbb{R}^3) : m_q(f) < \infty\}$ with $m_q(f) := \int_{\mathbb{R}^3} |v|^q f(dv)$.
- For $\theta \in (0, \pi/2)$, define $H(\theta) = \int_\theta^{\pi/2} \beta(x)dx$, it is a continuous decreasing bijection.
- For $z \in [0, \infty)$, denote $G(z) = H^{-1}(z)$ the inverse function of $H(\theta)$.
- For $z \in [0, \infty)$, $\varphi \in [0, 2\pi)$, $v, v_* \in \mathbb{R}^3$ and $K \in [1, \infty)$, $c(v, v_*, z, \varphi) = a[v, v_*, G(z/|v - v_*|^\gamma), \varphi]$.
- Two probability spaces: $(\Omega, \mathcal{F}, \mathbb{P})$, $([0,1], \mathcal{B}([0,1]), d\alpha)$.
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- For $z \in [0, \infty)$, $\varphi \in [0, 2\pi)$, $v, v_* \in \mathbb{R}^3$ and $K \in [1, \infty)$, $c(v, v_*, z, \varphi) = a[v, v_*, G(z/|v - v_*|^\gamma), \varphi]$.
- Two probability spaces: $(\Omega, \mathcal{F}, \mathbb{P})$, $([0,1], \mathcal{B}([0,1]), d\alpha)$.
- A stochastic process defined on the latter space is called an α-processes. Denote the expectation on $[0,1]$ by $\mathbb{E}_\alpha$ and the laws by $\mathcal{L}_\alpha$.

Some notation:

- $\mathcal{P}(\mathbb{R}^3)$: the set of probability measures on $\mathbb{R}^3$.
- For $q > 0$, $\mathcal{P}_q(\mathbb{R}^3) = \{f \in \mathcal{P}(\mathbb{R}^3) : m_q(f) < \infty\}$ with $m_q(f) := \int_{\mathbb{R}^3} |v|^q f(dv)$.
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Method: Tanaka's approach for the Maxwellian molecules

Goal: For moderately soft potentials i.e. $\gamma \in (-1, 0)$,

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Theorem 1 (C.Villani'98, Fournier-Mouhot'09)

Let $q \geq 2$ such that $q > \gamma^2/(\gamma+\nu)$. Let $f_0 \in \mathcal{P}_q(\mathbb{R}^3)$ have a density with $\int_{\mathbb{R}^3} f_0(v)|\log f_0(v)|dv < \infty$ and let $p \in (3/(3+\gamma), p_0(\gamma,\nu,q))$, with $p_0(\gamma,\nu,q) = \frac{q-\gamma}{q(3-\nu)/3-\gamma} \in (3/(3+\gamma), 3/(3-\nu))$.
Then, the Boltzmann equation has a unique weak solution $f \in L^\infty\big([0,\infty),\mathcal{P}_2(\mathbb{R}^3)\big) \cap L^1\big([0,\infty),L^p(\mathbb{R}^3)\big)$.

- Hard potentials: X. Lu, C. Mouhot, T. Elmroth, L. Desvillettes
- Maxwell molecules: G. Toscani and C. Villani
- Very soft potentials: C. Villani, N. Fournier and H. Guérin

Theorem 2 (Xu'16)

$(f_t)_{t\geq 0} \in L^\infty\big([0,\infty),\mathcal{P}_2(\mathbb{R}^3)\big) \cap L^1\big([0,\infty),L^p(\mathbb{R}^3)\big)$ is the unique weak solution given in Thm 1. Then for any other weak solution $(\tilde{f}_t)_{t\geq 0} \in L^\infty\big([0,\infty),\mathcal{P}_2(\mathbb{R}^3)\big)$, we have, for any $t \geq 0$,

$$\mathcal{W}_2^2(f_t,\tilde{f}_t) \leq \mathcal{W}_2^2(f_0,\tilde{f}_0)\exp\Big(C_{\gamma,p}\int_0^t (1+\|f_s\|_{L^p})ds\Big).$$

In particular, we have uniqueness for the Boltzmann equation when starting from f_0 in the class of all measure solutions in $L^\infty\big([0,\infty),\mathcal{P}_2(\mathbb{R}^3)\big)$.

Step 1: Associate a weak solution to some SDE with any weak solution.

Proposition 1 (Xu'16)

Consider any weak solution $(\tilde{f}_t)_{t\geq 0} \in L^\infty\big([0,\infty),\mathcal{P}_2(\mathbb{R}^3)\big)$. Then there exists, on some probability space, a random variable X_0 with law $\tilde{f}_0$, independent of a Poisson measure $M(ds,d\alpha,dz,d\varphi)$ on $[0,\infty)\times[0,1]\times[0,\infty)\times[0,2\pi)$ with intensity $dsd\alpha dzd\varphi$, a measurable family $(X_t^*)_{t\geq 0}$ of α-random variables and a càdlàg adapted process $(X_t)_{t\geq 0}$ solving

$$X_t = X_0 + \int_0^t\int_0^1\int_0^\infty\int_0^{2\pi} c\big(X_{s-},X_s^*(\alpha),z,\varphi\big)M(ds,d\alpha,dz,d\varphi) \quad (1)$$

and such that for all $t\geq 0$, $\mathcal{L}(X_t)=\mathcal{L}_\alpha(X_t^*)=\tilde{f}_t$.

It is a jump process, and the proof is quite complicated since $\tilde{f}$ does not have any regularity.

Step 2: It is hard to associate a stong solution to some SDE with any weak solution in $L^{\infty}\big([0,\infty),\mathcal{P}_2(\mathbb{R}^3)\big)\cap L^1\big([0,\infty),L^p(\mathbb{R}^3)\big)$ with some given φ_0. Hence, we consider a truncation version.

Proposition 2 (Xu'16)

$(f_t)_{t\geq 0}$ is the unique weak solution given in Thm 1. $W_0 \sim f_0$ (independent of M) satisfies $\mathbb{E}[|W_0 - X_0|^2] = \mathcal{W}_2^2(f_0, \tilde{f}_0)$ and, for each $t \geq 0$, an α-random variable W_t^* satifies $\mathcal{L}_\alpha(W_t^*) = f_t$ and $\mathbb{E}_\alpha[|W_t^* - X_t^*|^2] = \mathcal{W}_2^2(f_t, \tilde{f}_t)$. Then for $K \geq 1$, the equation

$$W_t^K = W_0 + \int_0^t \int_0^1 \int_0^K \int_0^{2\pi} c(W_{s-}^K, W_s^*(\alpha), z, \varphi + \varphi_{s,\alpha,K}) M(ds, d\alpha, dz, d\varphi), \quad (2)$$

with $\varphi_{s,\alpha,K} = \varphi_0(X_{s-} - X_s^*(\alpha), W_{s-}^K - W_s^*(\alpha))$, has a unique solution. Moreover, setting $f_t^K = \mathcal{L}(W_t^K)$ for each $t \geq 0$, it holds that for all $T > 0$,

$$\lim_{K\to\infty} \sup_{[0,T]} \mathcal{W}_2^2(f_t^K, f_t) = 0. \quad (3)$$

Step 3: Coupling these two processes.

- $\mathcal{W}_2^2(f_t,\tilde{f}_t) \le \mathcal{W}_2^2(f_t,f_t^K) + \mathcal{W}_2^2(f_t^K,\tilde{f}_t) \le \limsup_{K\to\infty} \mathbb{E}[|W_t^K - X_t|^2]$.
- $\mathbb{E}[|W_t^K - X_t|^2]$ by the Itô formula and some technics.
- $\limsup_{K\to\infty} \mathbb{E}[|W_t^K - X_t|^2] \le \mathcal{W}_2^2(f_0,\tilde{f}_0)\exp\left(C_{\gamma,p}\int_0^t (1+\|f_s\|_{L^p})ds\right)$ by the Gröwall lemma.

Thank you very much!